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Resumen

El estudio utilizo un modelo de finanzas de retornos en tiempo discreto para analizar si cambios en el precio del aceite de soja, principal insumo en la producción de biodiesel en EEUU afecta el de este último. Modelos empíricos de valoración de activos intentan utilizar precios observados para extraer información sobre variables latentes y parámetros estructurales Estos modelos, que en muchas ocasiones incluyen variables de estad de alta dimensionalidad pueden ser convenientemente estimados usando métodos bayesianos. Los resultados de este estudio indican que el precio del aceite de soja no tiene un gran impacto directo en el precio del biodiesel en el corto plazo, o en base diaria.

Palabras clave: Aceite de soja, biocombustibles, biodiesel, métodos Bayesianos, modelos de valoración de activos

Código JEL: C11, C13, C32, C51, G12, Q41

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Abstract

This work utilized a discrete time return model of finance to analyze whether prices changes of soybean oil, the main feedstock for biodiesel production in the US affect the prices of biodiesel. Empirical models of asset pricing attempt to extract information about latent state variables and structural parameters from observed prices. These models, which often involve high dimension latent state variables, can be conveniently estimated using Bayesian methods. Results from this study indicate the price of soybean oil does not have a strong direct impact on the price of biodiesel in the short run, or in a daily basis.

Key words: Soybean oil, biofuels, biodiesel, Bayesian methods, models of asset pricing

JEL Classification: C11, C13, C32, C51, G12, Q41

1. Introducción

The markets for biofuels in general and biodiesel in particular are heavily affected and driven by policies. The supporting policies are generally advocated on environmental, national and energy security, and rural development grounds. In fact, in the absence of public support very little quantities of biodiesel would be produced. The main reason is that vegetable oils, the main input in biodiesel production is simply more expensive than petro-diesel, the fuel being replaced. Feedstock (vegetable oils) costs usually account for between 80% and 90% of the total costs of production. The remainder is given mostly by energy, alcohol, catalysts, labor, marketing and other overhead expenses.

There are two major policy interventions that support biodiesel production, known as the "blender's credit" and the Renewable Fuels Standard (RFS) mandated consumption volumes. The blender's credit for biodiesel was first introduced in the American Jobs Creation Act of 2004 (Carriquiry 2007). Under the Act, blenders receive $1 dollar per gallons of biodiesel blended with diesel. The objective is to lower the cost of biodiesel to user to encourage demand. The first RFS was introduced in the Energy Policy Act of 2005 (Energy Bill of 2005), which mandates fuel producers to blend increasing quantities of renewable fuels, without distinguishing between different types of biofuels. The Energy Independence and Security Act of 2007 (EISA 2007) expanded the quantities that needed to be blended and carved out a specific market for biodiesel production, in what is known as RFS2. Under the RFS2, the quantities of biodiesel to be blended increased from 500 million gallons in 2009 to 1 billion gallons in 2012. Due to delays in implementation of the regulations, 2011 is the first year in which the mandate will be enforced.

In the absence of supporting policies, the price of biodiesel should be similar to that of diesel. Also, given the installed capacity to convert soybean oil into biodiesel, margins to biodiesel productions should not open beyond certain quantities that allow plants to earn a normal profit. If profitability exceeds this amount, production will be increased, raising the demand for feedstocks and their prices, reducing margins. Following this line, the price of soybean oil, the main feedstock for biodiesel production in the U.S. should be related to both the price of biodiesel and diesel fuel. In this work, we will focus on the relationship between soybean oil and biodiesel prices. As the variables that modulate the valuation of soybean oil as a major input for biodiesel production are not observable, we will utilize and estimate a model commonly used in the asset pricing literature, when latent variables are involved.

There is a large literature analyzing the relationship between the prices of energy and commodities.[1] The basic hypothesis is that given our current capacity to convert crops into biofuels, when the price of energy is high enough, the price of crops will be tied to their value as a source of energy. The bulk of this literature focuses on the relationship between crude oil or gasoline and corn. Some articles also include soybeans. However, to the best of our knowledge, much less attention has been paid to the relation between biodiesel prices and their feedstock.

Statistical analyses of the relationships between energy and food commodity prices have generally not found clear evidence on an impact of crude oil prices on the prices of food crops in general and oilseeds and vegetable oils in particular (Zilberman et al. 2012; Mallory, Hayes, and Irwin 2012; Saunders, Balagats, and Gruere, 2011). It should be noted that many of the studies reviewed by these authors included most of the data from periods in which biofuels, and in particular

[1] For recent reviews see Zilberman et al. (2012); Mallory, Hayes, and Irwin (2012).

biodiesel was not produced in significant quantities as to provide the direct linkages between the energy and crop markets these statistical studies attempt to detect. Zilberman et al. (2012) advance two plausible explanations of why the impact had been difficult to detect through prices. First, while, the work is attempting to assess the total price effect from using food commodities to fuel production, the estimated models capture the marginal effects. They also explain that the directional effect of the price of biofuel on the price of food commodity depends on the cause of the change in biofuel prices. However, there is little disagreement in that biofuel production levels increase commodity prices, which explains the broad support they receive from agricultural producers (Babcock 2011).

Empirical models of asset pricing attempt to extract information about latent state variables and structural parameters from observed prices. These models, which often involve high dimension latent state variables, can be conveniently estimated using Bayesian methods. These methods treat the parameters of the models as random variables, having a distribution, which depends on observed state variables X and dependent variables Y. The information is combined in the form to obtain a joint posterior distribution of the model parameters $p(\Theta | X, Y)$, which is used to simulate the individual parameters and obtain the statistics of interest about them. This method avoids the infeasible task of working out the integral for the latent variables which is needed for maximum likelihood methods.

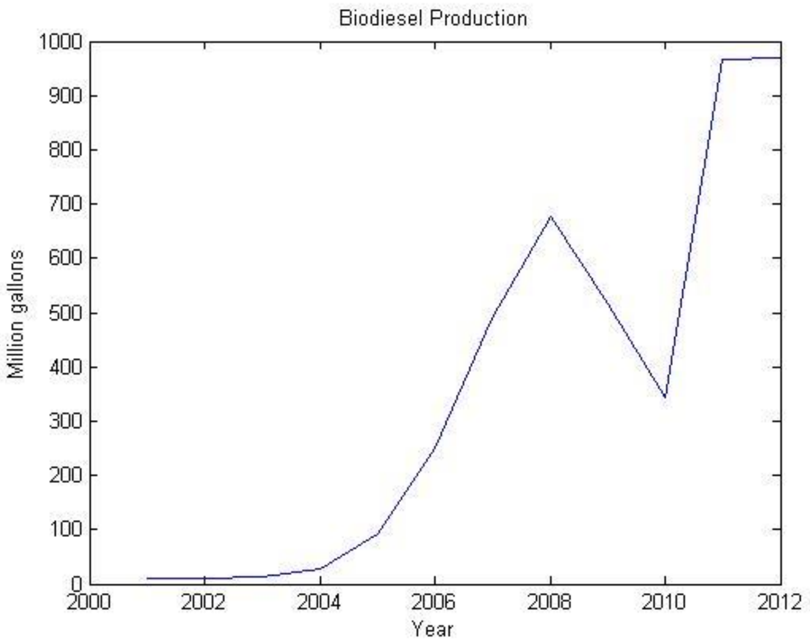
For this study, we adapt a popular discrete time return model of finance and use MCMC methods to estimate its parameters. We first conduct a simulation study to verify the method is able to identify the model parameters and latent variables. We then illustrate a potential application of the method by studying the relationship between biodiesel and soybean oil prices.

The paper is organized as follows. In section 2, we provide some background on the biodiesel market and the expected relationship between biodiesel and soybean oil prices. Section 3 introduces the model for pricing dynamics. Section 4 provides an overview of MCMC methods for estimating the model. Section 5 specifies the likelihood to the used, the parameter prior, and derive the joint posterior distribution for the parameters. Simulations to assess the ability of the method to uncover parameters of the model are performed in Section 6. Section 7 provides the empirical application and conclusions are presented in Section 8.

2. Recent developments in the market for biodiesel

Biodiesel production has grown rapidly in the last few years, as mentioned mostly as a result of a favorable policy environment. While production was negligible up to 2004, it grew rapidly to reach almost 1 billion gallons in both 2011 and 2012. US production is expected to keep growing in 2013 (Energy Information Administration 2013). The decline in 2010 can be directly mapped to the expiration of a subsidy providing $1 per gallon of biodiesel blended into diesel fuels, which was reinstated in 2011, together with the implementation of the RFS. While the credit was let expire again in 2012, the mandated consumption level under RFS (1 billion gallons) provided enough incentives for producers to reach that production level despite expiration of the credit. For completeness, it should be noted the blender’s credit was reinstate in early 2013, and was made retroactive to 2012.

Figure 1: Recent evolution of biodiesel production levels in the U.S.



Source: Elaborated based on Energy Information Administration data.

Soybean oil is the main feedstock used for biodiesel production in the US (see Table 1). This is especially the case in 2011 and 2012, when higher volumes of biodiesel were produced. There is simply not enough of other fats and oils to support higher production volumes. Given that soybean oil is the main feedstock in biodiesel production, and that feedstock costs account for between 80% and 90% of costs of production, it is hypothesized here that changes in the price of soybean oil have an impact on the prices of biodiesel.

Table 1. Feedstock Used for Biodiesel Production in the US

Year	Vegetable oil		Animal Fats	Recycled feeds	Other	Total
	Other	Soybeans				
			million pounds			
2010	358	1,141	645	286	33	2,463
2011	1,151	4,153	1,289	666	27	7,286
2012	1,358	4,023	1,010	900	1	7,292
			% of total			
2010	15%	46%	26%	12%	1%	100%
2011	16%	57%	18%	9%	0%	100%
2012	19%	55%	14%	12%	0%	100%

Source: Elaborated based on Energy Information Administration data.

3. A Model

The return dynamics model including the latent variable is introduced in this section, together with the joint distribution of the dependent and latent variables. Let y_{t+1} be the price $t+1$, and $\mu_t = E\left(y_{t+1}\right)$ the expected price in period $t+1$. The returns dynamic model that will be used here can be written as

(1)
$$\begin{cases} y_{t+1} = \mu_t + \sigma_y \varepsilon_{t+1}^y \\ \mu_{t+1} = E_\mu + \beta\left(\mu_t - E_\mu\right) + \gamma Z_{t+1} + \sigma_\mu \varepsilon_{t+1}^\mu \end{cases},$$

where the error terms ε_{t+1}^y and ε_{t+1}^μ are $N\left(0,1\right)$ distributed, with $corr(\varepsilon_{t+1}^y, \varepsilon_{t+1}^\mu) = \rho$, E_μ is the unconditional mean of the latent variable μ_t, and Z_t denotes an explanatory variable hypothesized to affect the change in prices through the latent variable. Thus, the model contains the observed $Y = \left\{y_t\right\}_{t=0}^T$, $Z = \left\{Z_t\right\}_{t=0}^T$, latent expected return variables $\underline{\mu} = \left\{\mu_t\right\}_{t=0}^T$, and parameters $\Theta = \left\{E_\mu, \beta, \sigma_y, \sigma_\mu, \rho, \gamma\right\}$ to be estimated.

The estimation of the parameters of the model will use Bayesian methods to obtain posterior distributions for both $\underline{\mu}$ and Θ, conditional on the data. That is, we need to obtain the joint posterior distribution $p(\underline{\mu}, \Theta | Y)$. To obtain this distribution, a likelihood for the data in the model, and prior distributions for the parameters are needed. We focus first on the likelihood component. The choice of priors and the derivations of the posterior are presented later.

Conditioning on the latent variables, the observed explanatory variables and model parameters, changes in the joint distribution of the latent variables and prices $p\left(y_{t+1}, \mu_{t+1} \middle| \mu_t, \Theta, Z_{t+1}\right)$ follow a bivariate normal distribution given by

$$\left.\begin{pmatrix} y_{t+1} \\ \mu_{t+1} \end{pmatrix}\right| \mu_t, Z_t, \Theta \sim N\left(\begin{bmatrix} \mu_t \\ E_\mu + \beta(\mu_t - E_\mu) + \gamma Z_{t+1} \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \rho\sigma_y\sigma_\mu \\ \rho\sigma_y\sigma_\mu & \sigma_\mu^2 \end{bmatrix}\right)$$

And the joint distribution can be written as

$$p\left(Y,\underline{\mu},\Theta\right)=\prod_{t=0}^{T-1}p(y_{t+1},\mu_{t+1}\left|\mu_t,\Theta,Z_{t+1}\right.)\Pi\left(\Theta\right)$$

$$=\prod_{t=0}^{T-1}\frac{1}{\sqrt{2\pi}\sigma_\mu\sigma_y\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left\{\begin{array}{l}\left(\frac{y_{t+1}-\mu_t}{\sigma_y^2}\right)^2\\-2\rho\left(\frac{y_{t+1}-\mu_t}{\sigma_y}\right)\left(\frac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right)+\\\left(\frac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right)^2\end{array}\right\}\right]$$

$$\times p\left(E_\mu\right)p\left(\beta\right)p\left(\sigma_y\right)p\left(\sigma_\mu,\rho\right)p\left(\gamma\right)$$

where $p\left(E_\mu\right)$, $p\left(\beta\right)$, $p\left(\sigma_y\right)$, $p\left(\sigma_\mu,\rho\right)$, and $p\left(\gamma\right)$ are prior distributions for the parameters to be specified next. Posterior distributions for the parameters and for $\underline{\mu}$ will be obtained from this distribution. For convenience, we select conjugate priors and priors that will result in closed forms for the posterior distributions of most of the parameters. While this simplifies the numerical simulations this is not strictly needed, as an array of techniques (e.g., Gibbs sampling) allows us to sample from the posteriors even without knowing their full functional form. The priors will be specified with large variance ("uninformative") so as to have minimal influence on the results.

The following priors were defined for the parameters.

- $E_\mu\sim N\left(e,E^2\right)$
- $\beta\sim N(f,F^2)$ where $-1<\beta<1$
- $\frac{1}{\sigma_y^2}\sim Gamma(\alpha,\beta^*)$ with $\sigma_y^2>0$.
- $\gamma\sim N(g,G^2)$
- For (σ_μ^2,ρ), to simplify the algorithm and facilitate convergence, we use the reparameterization (see Jacquier, Polson and Rossi 1994)

$$\begin{cases}\phi_\mu=\sigma_\mu\rho\\w_\mu=\sigma_\mu^2(1-\rho^2)\end{cases}$$

with the additional assumption

$$\begin{array}{c}\phi_\mu\left|w_\mu\right.\sim N\left(0,\frac{1}{2}w_\mu\right)\\w_\mu\sim Inv-Gamma\left(a,b\right)\end{array}.$$

Putting all together, the joint distribution is given by

$$p\left(Y,\underline{\mu},\Theta\right)=\prod_{t=0}^{T-1}p(y_{t+1},\mu_{t+1}\left|\mu_t,\Theta,Z_{t+1}\right.)\Pi\left(\Theta\right)=$$

$$\prod_{t=0}^{T-1}\frac{1}{\sqrt{2\pi\sigma_\mu\sigma_y\sqrt{1-\rho^2}}}\exp\left[-\frac{1}{2(1-\rho^2)}\left\{\begin{array}{l}\left(\dfrac{y_{t+1}-\mu_t}{\sigma_y^2}\right)^2\\ -2\rho\left(\dfrac{y_{t+1}-\mu_t}{\sigma_y}\right)\left(\dfrac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right)+\\ \left(\dfrac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right)^2\end{array}\right\}\right]$$

$$\times\frac{1}{\sqrt{2\pi E}}\exp\left[-\frac{\left(E_\mu-e\right)^2}{2E^2}\right]\frac{1}{\sqrt{2\pi F}}\exp\left[-\frac{\left(\beta-f\right)^2}{2F^2}\right]$$

$$\times\left(\frac{1}{\sigma_y^2}\right)^{-1}\frac{\left(\beta^*\right)^\alpha\exp(-1/(\beta^*\sigma_y^2))}{\Gamma\left(\alpha\right)}\left(\frac{1}{w_\mu}\right)^{a+1}\exp\left[-\frac{1}{bw_\mu}\right]$$

$$\times\frac{1}{\sqrt{\pi w_\mu}}\exp\left[-\frac{\phi_\mu^2}{w_\mu}\right]\frac{1}{\sqrt{2\pi G}}\exp\left[-\frac{\left(\gamma-g\right)^2}{2G^2}\right]$$

which can be used to derive the posterior distributions of interest. The detailed derivation of the distributions is presented in the Appendix. Before showing the form of the posterior distributions, we present the estimation method that will allow us to estimate the parameters of this model.

4. Estimation Methods

For complicated models like the one used here, sampling directly from the joint posterior distribution is not usually workable. For these situations, methods based on Markov chain simulations like the Gibbs sampler or applications of the Metropolis algorithm are commonly used. A combination of the Gibbs sampler and the Metropolis-Hasting algorithm are used in this study. Before explaining their implementation in the current context, I provide a brief overview of the logic behind Markov chain simulations.

A Markov chain is a model for a stochastic process, whose states evolve according to a transition probability and depend (for a first order chain) only on the most recent state $\left(P\left(X_t\left|X_{t-1},\ldots,X_0\right.\right)=P\left(X_t\left|X_{t-1}\right.\right)\right)$. The idea is to simulate random samples that converge to a stationary target distribution (Gelman et al. 2010). In applications, this target distribution is the joint posterior distribution of the parameters of the model we want to estimate. A key is to run the simulations long enough so that the draws are close to what would have resulted from sampling from the joint posterior distribution directly. In this application, the objective is to obtain samples to characterize moments of the distributions of the parameters Θ and latent variables conditioned on the observed data $\{y_t\}_{t=0}^{T}$ and covariates $\{Z_t\}_{t=0}^{T}$. The MCMC to be implemented will sample from these high dimensional distribution. MCMC algorithms are based on the insight that the complete conditional distributions characterize the joint distribution. In other words, we know that $p(\Theta, \underline{\mu} | Y, Z)$ can be characterized using $p(\Theta | \underline{\mu}, Y, Z)$ and $p(\underline{\mu} | \Theta, Y, Z)$ together.

For some initial values $\Theta^{(0)}$ and $\underline{\mu}^{(0)}$, and given the conditional distributions we can draw $\underline{\mu}^{(1)} \sim p(\underline{\mu} | \Theta^{(0)}, Y, Z)$ and then $\Theta^{(1)} \sim p(\Theta | \underline{\mu}^{(1)}, Y, Z)$. The updated value of Θ can then be used to update $\underline{\mu}$ and the process is iterated a sufficiently large number of times. In this way the algorithm generates a sequence random variables whose distribution converges to the target distribution $p(\Theta, \underline{\mu} | Y, Z)$.

If we can get the complete conditional distributions, the Gibbs or Successive Substitution sampling can be directly used. The Gibbs algorithm works as follows. For some vector of starting values $\Theta^0 = (\theta_1^0, ..., \theta_k^0)$, the Gibbs sampler generates;

- a value θ_1^1 from $\theta_1^1 \sim h(\theta_1^0, \theta_2^0, ..., \theta_k^0)$
- a value θ_2^1 from $\theta_2^1 \sim h(\theta_1^1, \theta_2^0, ..., \theta_k^0)$
- a value θ_3^1 from $\theta_3^1 \sim h(\theta_1^1, \theta_2^1, \theta_3^0, ..., \theta_k^0)$
- until the last variable to be sampled for this iteration θ_k^1 from $\theta_k^1 \sim h(\theta_1^1, \theta_2^1, ..., \theta_k^0)$.
- Return to generating the first variable.

Repeating this process a large enough number of times N, we expect that approximately $\Theta^N \sim h$. The theoretical properties of the posterior distribution can then be approximated through the sample properties of $\{\Theta^{B+1}, \Theta^{B+2}, ..., \Theta^N,\}$, where B is the burn-in or initial number of observation (pre-convergence) discarded.

While we can use the Gibbs algorithm for most variables in this model, not all conditional distributions have a known standard form from which we can sample directly. In this case, we can use other sampling algorithms like the Metropolis-Hasting algorithm. The Metropolis-Hasting algorithm works as follows. For a target distribution $p(\theta | y)$, we need to find a proposed distribution $h(\theta)$ from which we know how to sample. Two conditions need to be satisfied by this proposed distribution; a) $h(\theta) = 0$ implies $p(\theta | y) = 0$, and $p(\theta)/h(\theta) < M$ a known finite upper bound for all θ. The algorithm can be implemented as follows. At the $(j+1)^{th}$ iteration:

- Generate a proposed value $\theta^{j*} \sim h(\bullet | \theta^{(j-1)})$

- Compute $r_j = \frac{p(\theta^{j*}) / h(\theta^{j*} | \theta^{j-1})}{p(\theta^{j-1}) / h(\theta^{j-1} | \theta^{j*})}$ and generate $S_j \sim Bernoulli(\min(1, r_j))$
- Take $\theta^j = S_j \theta^{j*} + (1 - S_j)\theta^{j-1}$.

The samples obtained from these algorithms can then be used to estimate the parameters and latent variables using the Monte Carlo method, as the posterior mean of $p(\Theta, \underline{\mu})$. This posterior mean can be approximated by $(1/T)\sum_{t=1}^{T}\Theta^t$ and $(1/T)\sum_{t=1}^{T}\underline{\mu}^t$.

5. Derivation of the Posterior Distributions of the Parameters and Latent Variables

This section describes the Bayesian MCMC methods that were used to estimate this model. As stated above, the high dimensionality of the latent variables complicates the estimation. From a computational standpoint, it is almost impossible to integrate the large number of latent variables as required to implement either likelihood or method of moments approaches. Bayesian methods allow us to by-pass these problems. In particular, Bayesian methods have been shown to perform well in this type of problems (see e.g., Du, Hayes, and Yu 2011). The applied MCMC algorithm generates samples by iteratively drawing from the derived conditional posteriors, which are fully described in the Appendix.

Monte Carlo Markov Chains were used to sample from the posterior distributions of the parameters specified below in order to characterize their distributions and be able to estimate their expected values and variances.

Posterior distribution of β ***conditioned on*** E_μ , σ_y , σ_μ , ρ , $\underline{\mu}$, γ , Y ***, and*** Z ***:***

$$\beta | E_\mu, \sigma_y, \sigma_\mu, \rho, \gamma, Y, Z \sim N\left(\frac{S}{W}, \frac{1}{W}\right)$$

where $$S = \frac{-\rho}{1-\rho^2}\sum_{t=1}^{T-1}\frac{C_{t+1}(\mu_t - E_\mu)}{\sigma_y \sigma_\mu} + \frac{1}{1-\rho^2}\sum_{t=1}^{T-1}\frac{D_{t+1}(\mu_t - E_\mu)}{\sigma_\mu^2} + \frac{f}{F^2},$$

$W = \frac{1}{(1-\rho^2)}\sum_{t=1}^{T-1}\left(\frac{\mu_t - E_\mu}{\sigma_\mu}\right)^2 + \frac{1}{F^2}$, $C_{t+1} = y_{t+1} - \mu_t$, and $D_{t+1} = \mu_{t+1} - E_\mu - \gamma Z_{t+1}$. Notice however that the prior for beta is a truncated normal. The posterior is also truncated, and the truncation is expressed as $\beta | \Theta, Y, Z \sim N\left(\frac{S}{W}, \frac{1}{W}\right) * 1_{(-1,1)}$.

Posterior distribution of E_μ ***conditioned on*** β ***,*** σ_y ***,*** σ_μ ***,*** ρ ***,*** $\underline{\mu}$ ***,*** γ ***,*** Y ***, and*** Z ***:***

$$E_\mu \left| \beta, \sigma_y, \sigma_\mu, \rho, \gamma, Y, Z \sim N\left(\frac{S}{W}, \frac{1}{W}\right)\right.$$

Where $S = \frac{-\rho}{1-\rho^2}\sum_{t=1}^{T-1}\frac{C_{t+1}(1-\beta)}{\sigma_y\sigma_\mu} + \frac{1}{1-\rho^2}\sum_{t=1}^{T-1}\frac{D_{t+1}(1-\beta)}{\sigma_\mu^2} + \frac{e}{E^2}$,

$W = \frac{1}{(1-\rho^2)}\sum_{t=1}^{T-1}\left(\frac{1-\beta}{\sigma_\mu}\right)^2 + \frac{1}{E^2}$, $C_{t+1} = y_{t+1} - \mu_t$, and $D_{t+1} = \mu_{t+1} - \beta\mu_t - \gamma Z_{t+1}$.

Posterior distribution of γ ***conditioned on*** β ***,*** σ_y ***,*** σ_μ ***,*** ρ ***,*** $\underline{\mu}$ ***,*** E_μ ***,*** Y ***, and*** Z ***:***

$$\gamma \left| \beta, \sigma_y, \sigma_\mu, \rho, E_\mu, Y, Z \sim N\left(\frac{S}{W}, \frac{1}{W}\right)\right.$$

Where $S = \frac{-\rho}{1-\rho^2}\sum_{t=1}^{T-1}\frac{C_{t+1}Z_{t+1}}{\sigma_y\sigma_\mu} + \frac{1}{1-\rho^2}\sum_{t=1}^{T-1}\frac{D_{t+1}Z_{t+1}}{\sigma_\mu^2} + \frac{r}{R^2}$, $W = \frac{1}{(1-\rho^2)}\sum_{t=1}^{T-1}\left(\frac{Z_{t+1}}{\sigma_\mu}\right)^2 + \frac{1}{R^2}$,

$C_{t+1} = y_{t+1} - \mu_t$, and $D_{t+1} = \mu_{t+1} - E_\mu - \beta(\mu_t - E_\mu)$.

Posterior distribution of $\underline{\mu}$ ***conditioned on*** β ***,*** σ_y ***,*** σ_μ ***,*** ρ ***,*** γ ***,*** E_μ ***,*** Y ***, and*** Z ***:***

For the latent vector $\underline{\mu}$, the posterior distribution depends on the place in the series. Each element is updated once at a time. In particular, we need to derive the posteriors for μ_0 at $t = 0$ as μ_0 depends on μ_1, μ_t at $1 \le t \le T-1$ as μ_t depends on μ_{t-1} and μ_{t+1}, and μ_T at $t = T$ as μ_T depends on μ_{T-1}.

$\underline{\mu_0}$

$$\mu_0 \left| \beta, \sigma_y, \sigma_\mu, \rho, \gamma, E_\mu, Y, Z \sim N\left(\frac{S}{W}, \frac{1-\rho^2}{W}\right)\right.$$

where $S = \frac{y_1}{\sigma_y^2} - \frac{\rho y_1\beta}{\sigma_y\sigma_\mu} - \frac{\rho(\mu_1 - E_\mu - \gamma Z_1)}{\sigma_y\sigma_\mu} - \frac{\rho\beta E_\mu}{\sigma_y\sigma_\mu} + \frac{\mu_1\beta}{\sigma_\mu^2} - \frac{\mu_1 E_\mu}{\sigma_\mu^2} + \frac{\beta^2 E_\mu}{\sigma_\mu^2} - \frac{\beta\gamma Z_1}{\sigma_\mu^2}$,

$W = \frac{1}{\sigma_y^2} - \frac{2\rho\beta}{\sigma_y\sigma_\mu} + \frac{\beta^2}{\sigma_\mu^2}$.

$\underline{\mu_T}$

$$\mu_T \left| \beta, \sigma_y, \sigma_\mu, \rho, \gamma, E_\mu, Y, Z \sim N\left(\frac{S}{W}, \frac{1-\rho^2}{W}\right)\right.$$

where $S = \frac{\rho y_T}{\sigma_y \sigma_\mu} - \frac{\rho \mu_{T-1}}{\sigma_y \sigma_\mu} + \frac{E_\mu}{\sigma_\mu^2} + \frac{\beta \mu_{T-1}}{\sigma_\mu^2} - \frac{\beta E_\mu}{\sigma_\mu^2} + \frac{\gamma Z_T}{\sigma_\mu^2}$, $W = \frac{1}{\sigma_\mu^2}$.

$\underline{\mu}_t$ where $0 < t < T$

$$\mu_t \left| \beta, \sigma_y, \sigma_\mu, \rho, \gamma, E_\mu, Y, Z \sim N\left(\frac{S}{W}, \frac{1-\rho^2}{W}\right)\right.$$

where

$$S = \frac{y_{t+1}}{\sigma_y^2} + \frac{\rho\left(E_\mu - y_{t+1}\beta - \mu_{t+1} - \beta E_\mu - \gamma Z_{t+1} + y_t - \mu_{t-1}\right)}{\sigma_y \sigma_\mu} + \frac{\left(\mu_{t+1} - E_\mu\right)\beta + \beta^2 E_\mu - \beta\gamma Z_{t+1} + E_\mu + \beta\left(\mu_{t-1} - E_\mu\right) + \gamma Z_t}{\sigma_\mu^2}$$

, $W = \frac{1}{\sigma_y^2} - \frac{2\rho\beta}{\sigma_y \sigma_\mu} + \frac{\beta^2 + 1}{\sigma_\mu^2}$.

Posterior joint distribution of $\left(\rho, \sigma_\mu\right)$ ***conditioned on*** β **,** σ_y**,** $\underline{\mu}$ **,**γ**,** E_μ **,** Y **, *and*** Z **:**

$$\begin{cases} \phi_\mu \left| \omega_\mu \sim N\left(\frac{S}{W}, \frac{\omega_\mu}{W}\right)\right. \\ \omega_\mu \sim IG\left(\frac{T}{2} + a, \frac{1}{\frac{\sum_{t=1}^{T} D_t^2}{2} + \frac{1}{b} - \frac{S^2}{2W}}\right) \end{cases}$$

where $S = \sum_{t=1}^{T-1} C_t D_t$, $W = \sum_{t=1}^{T-1} C_t^2 + 2$, $C_t = y_{t+1} - \mu_t$, and

$D_t = \mu_{t+1} - E_\mu - \beta\left(\mu_t - E_\mu\right) - \gamma Z_{t+1}$.

Posterior distribution of σ_y ***conditioned on*** β **,**σ_μ **,**ρ **,** $\underline{\mu}$ **,** E_μ **,** γ**,** Y **, *and*** Z **:**

There is no closed form solution for the distribution of σ_y. The Metropolis-Hasting algorithm will be used to obtain samples from this distribution.

6. Simulation Study

To assess the ability of the used methods to estimate the parameters of the model, a simulations study is conducted here. For this purpose 30 sample paths of length $T = 1000$ where constructed using the known coefficients, hyperparameters for the priors and the assumed functional forms. All the simulations and model estimation were done using Matlab.

By our modeling assumptions (see Section 3), Equation (1) and the known coefficients can be used to simulate the data as

$$\begin{cases} y_{t+1} = \mu_t + \sigma_y \varepsilon_{t+1}^y \\ \mu_{t+1} = E_\mu + \beta\left(\mu_t - E_\mu\right) + \gamma Z_{t+1} + \sigma_\mu \varepsilon_{t+1}^\mu \end{cases},$$

with ε_{t+1}^y and ε_{t+1}^μ being correlated $\left(\rho = corr\left(\varepsilon_{t+1}^y, \varepsilon_{t+1}^\mu\right)\right)$ standard normal random variables. To obtain the correlated draws the following procedure was used. Two independent standard normal random variables (x, x_1) were simulated, and the variable $x_2 = \sqrt{1-\rho^2}\,x + \rho x_1$ calculated. Using this procedure we know that (x_1, x_2) are standard normal random variables with correlation ρ.[2] The exogenous series (Z_t) was also simulated for this exercise (using a normal distribution with mean zero and arbitrary variance of 0.90). The parameters used to simulate the data are included in Table 2.

Once the data is simulated, the MCMC algorithm described in section 4 is used to simulate from the posterior distributions of the parameters. Hyperparameters for the priors and starting values for the parameters are needed to initialize the algorithm. The starting values used for the parameters (with the exception of the initial vector for the latent variable $\underline{\mu}^{(0)}$) are also presented in Table 2. The data was used to initialize the vector a latent variables as $\underline{\mu}^{(0)} = \left(y_1, y_2, \ldots, y_{T-1}, y_T, y_T\right)$. The following hyperparameters for the priors were used *e=0, E=1, f=0, F=1,* $\alpha = 100$, $\beta^* = 2$, *a=2, b=200, r=0, R=1.*

For each sample path sequences of the random variables $\left\{E_\mu^{(g)}, \beta^{(g)}, \sigma_y^{(g)}, \sigma_\mu^{(g)}, \rho^{(g)}, \gamma^{(g)}\right\}_{g=1}^{G}$ and $\underline{\mu}^{(g)}$, with $G = 50,000$ were obtained using the MCMC algorithm. After a burn-in of the first 10,000 simulated values, the means of the posterior samples were used to estimate the parameters of the model. These posterior means calculated over the 30 sample paths are presented in Table 2, together with their true values, initial values, and the root mean square error (*RSME*) of the estimated values.

[2] $\operatorname{cov}\left(x_1, \sqrt{1-\rho^2}\,x + \rho x_1\right) = \sqrt{1-\rho^2}\operatorname{cov}\left(x_1, x\right) + \rho \operatorname{cov}\left(x_1, x_1\right) = \rho$

Table 2. Results from the simulation study. 30 samples of length 1000 were simulated. 50000 iterations were run, with a burn in of 10000.

	E_μ	β	σ_y	σ_μ	ρ	γ
Initial value	0.05	0.5	1	1	0.5	0.1
True parameter	0.1	0.9	0.15	0.1	-0.5	0.5
Posterior mean	0.0988	0.8997	0.1551	0.0987	-0.5182	0.4991
RSME of posterior	0.0475	0.0038	0.0246	0.0119	0.1092	0.0092

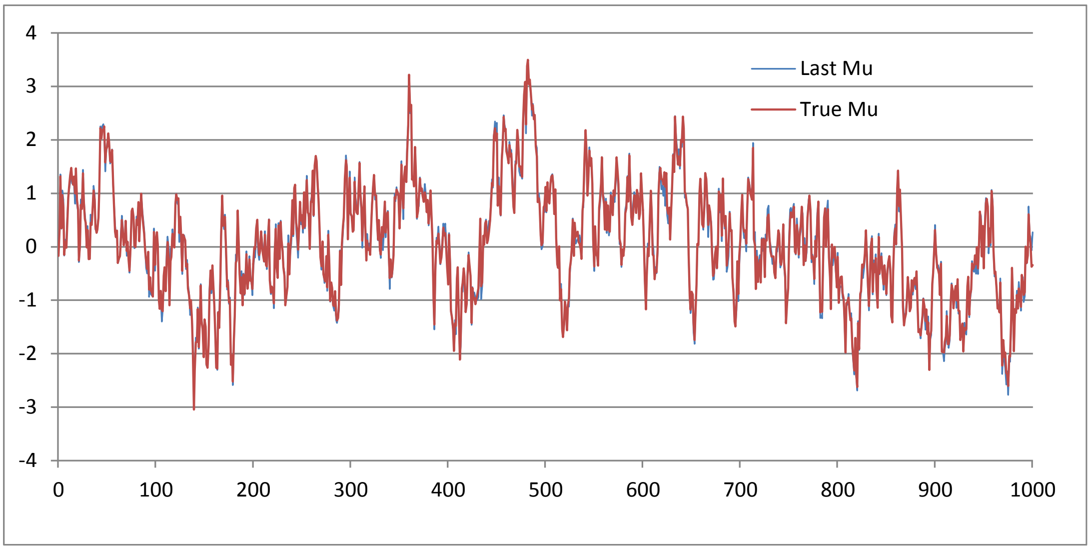
The table shows the method used here is able to accurately recover the parameters used to simulate the data. Plots of the chains, after the burn-in period are presented in the Appendix. These plots show all the chains converge. An analysis based on the results from a single sample path is presented next. This is the situation that will likely be encountered in most applications of the model, including the one presented here. Results of this run are summarized in Table 3. As shown in the table, not much accuracy is lost, with RSME of posteriors only increasing (as expected) slightly.

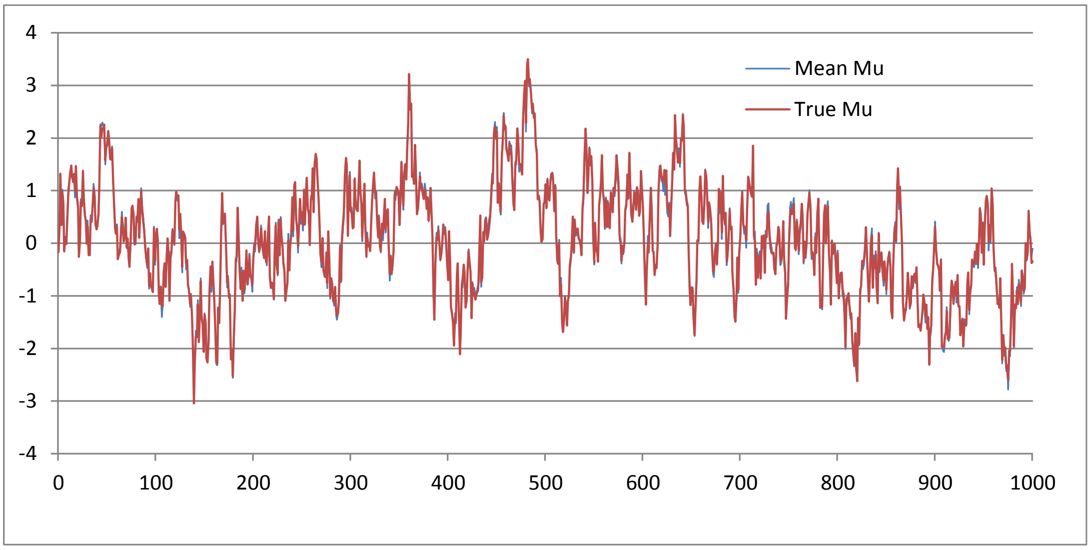
Table 3. Results from the simulation study for 1 sample of length 1000. 50000 iterations were run, with a burn in of 10000.

	E_μ	β	σ_y	σ_μ	ρ	γ
Initial value	0.05	0.5	1	1	0.5	0.1
True parameter	0.1	0.9	0.15	0.1	-0.5	0.5
Posterior mean	0.143	0.9028	0.1599	0.0903	-0.5194	0.4909
RSME of posterior	0.051	0.0039	0.0256	0.0134	0.1117	0.0108

Of interest is also whether the estimation procedure is able to accurately estimate the high dimensional vector of latent variables $\underline{\mu}$. Figure 2 shows that this is indeed the case. The upper panel of the figure plots the true value of the latent variables against the estimated value from the last iteration. The lower panel compares the true value against the mean of the last 100 iterations. Both charts show that the methods used here can accurately capture the latent variables. Once established the usefulness of the method, it is put to work in an application on the real data set, to answer the question of interest.

Figure 2. Latent vector $\underline{\mu}$ and estimates obtained from the MCMC methods



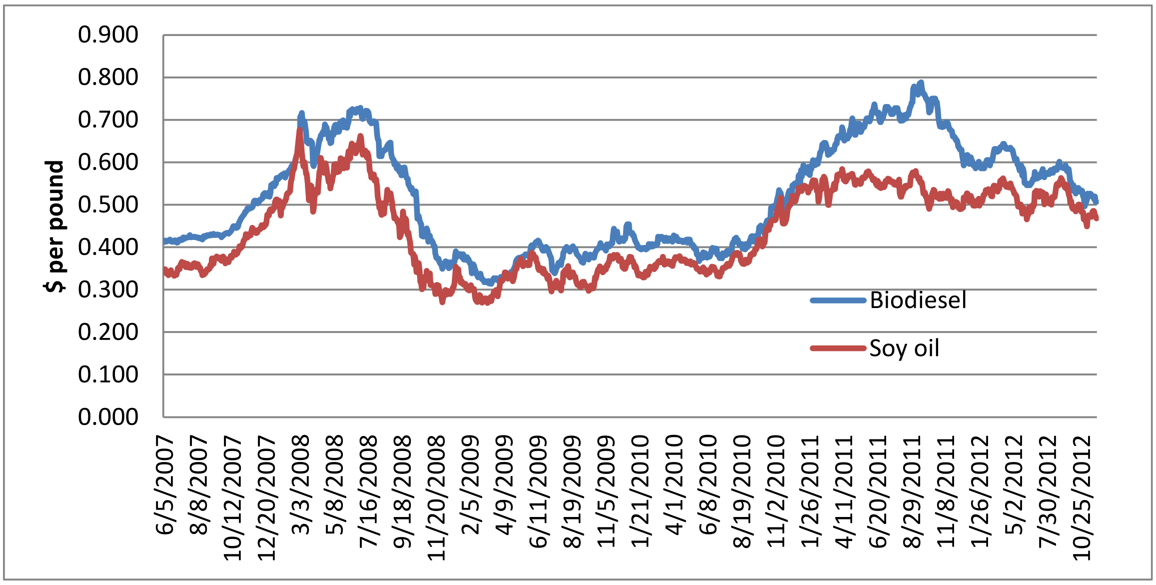


7. Application

The model presented was used to analyze the relationship between the prices of biodiesel in the Midwest, and its main feedstock, soybean oil. In particular, we wanted to assess whether changes in the price of soybean oil would anticipate short-term movement in the price of biodiesel. For this purpose, daily prices for the period June-2007 to December-2012 were used. After removing missing values, there were a total of 1,182 pairs of prices. The source of biodiesel price data is BiofuelsConnet. Soybean oil prices were obtained from the Chicago Mercantile Exchange.

The series are presented in Figure 3. It is clear from the figures that the prices of soybean oil and biodiesel follow similar patterns, both increasing and decreasing in similar periods. The work presented here formally analyzes whether changes in the prices of soybean oil lead, or precede changes in the prices of biodiesel. Of particular interest is the coefficient γ . A non-zero value for the parameter will indicate the prices of soybean oil are likely to affect those of biodiesel.

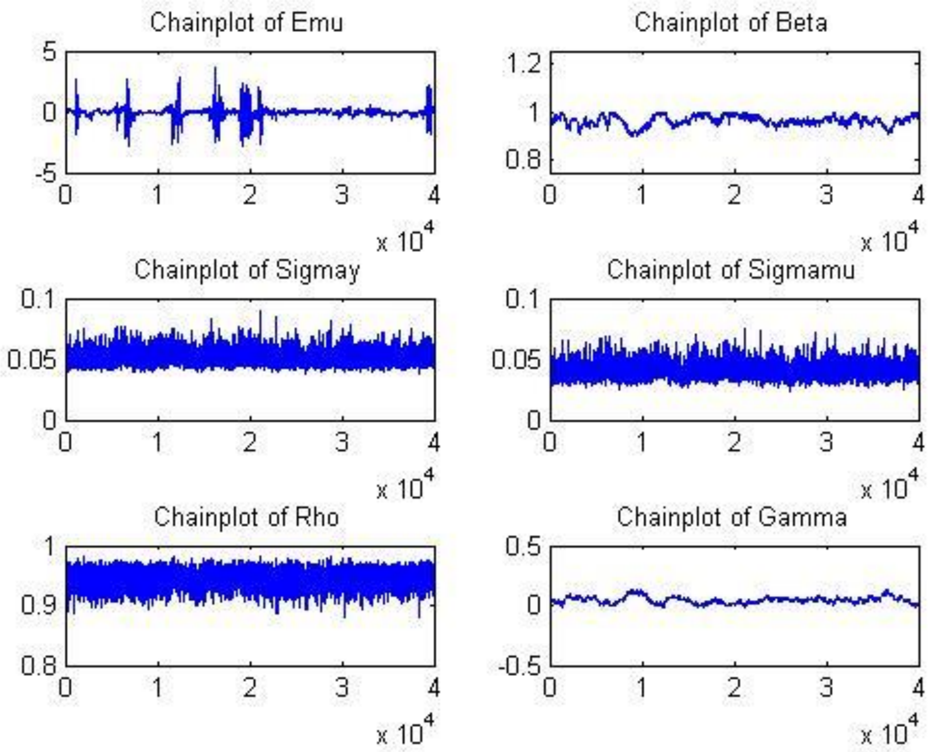
Figure 3. Daily prices of biodiesel and soybean oil



Notes: The biodiesel price is soy methyl esther price in the Midwest. Source: Biodiesel from BiofuelsConnect; Soybean oil from the Chicago Mercantile Exchange.

The model was estimated using the MCMC methods described above, running 150,000 iterations, and a burn-in of 110,000 iterations. Plots of the chains after the burn-in period are presented in Figure 4. It indicates the chains for all the parameters have converged and are stable.

Figure 4. Plots of the chains for all parameters of the model (150,000 iterations, and a burn-in of 110,000 iterations, first set of starting values for the parameters (presented in Table 4, below)).



Summary of the Simulations

The tables below (4 and 5) show summary statistics for the simulated parameters for two different sets of starting values. The estimated values for the means, standard deviation, and selected quantiles are similar for both sets of starting values. While the plots of the chains for the parameters obtained from the first set of starting values are presented above (Figure 4), the equivalent plots for the second set of starting values of the parameters are presented in the Appendix.

Table 4. MCMC estimates of the model parameters and quantiles of the posterior distributions of the parameters (first set of starting values for the parameters)

	E_μ	β	σ_y	σ_μ	ρ	γ
Starting point	1.0E-04	0.5	1.0	1.0	0.5	0.1
Mean of posterior	-0.044	0.964	0.045	0.037	0.930	0.058
std of posterior	0.335	0.020	0.005	0.005	0.015	0.037
2.5% Quantile of posterior	-0.774	0.917	0.040	0.029	0.899	-0.002
97.5% Quantile of posterior	0.629	0.998	0.060	0.051	0.960	0.125

Table 5. MCMC estimates of the model parameters and quantiles of the posterior distributions of the parameters (second set of starting values for the parameters)

	E_μ	β	σ_y	σ_μ	ρ	γ
Starting point	1.0E-04	0.1	1.0	1.2	0.1	0.5
Mean of posterior	-0.077	0.952	0.052	0.041	0.946	0.062
std of posterior	0.268	0.028	0.005	0.006	0.013	0.037
2.5% Quantile of posterior	-0.655	0.900	0.043	0.031	0.918	0.002
97.5% Quantile of posterior	0.402	0.997	0.063	0.054	0.967	0.133

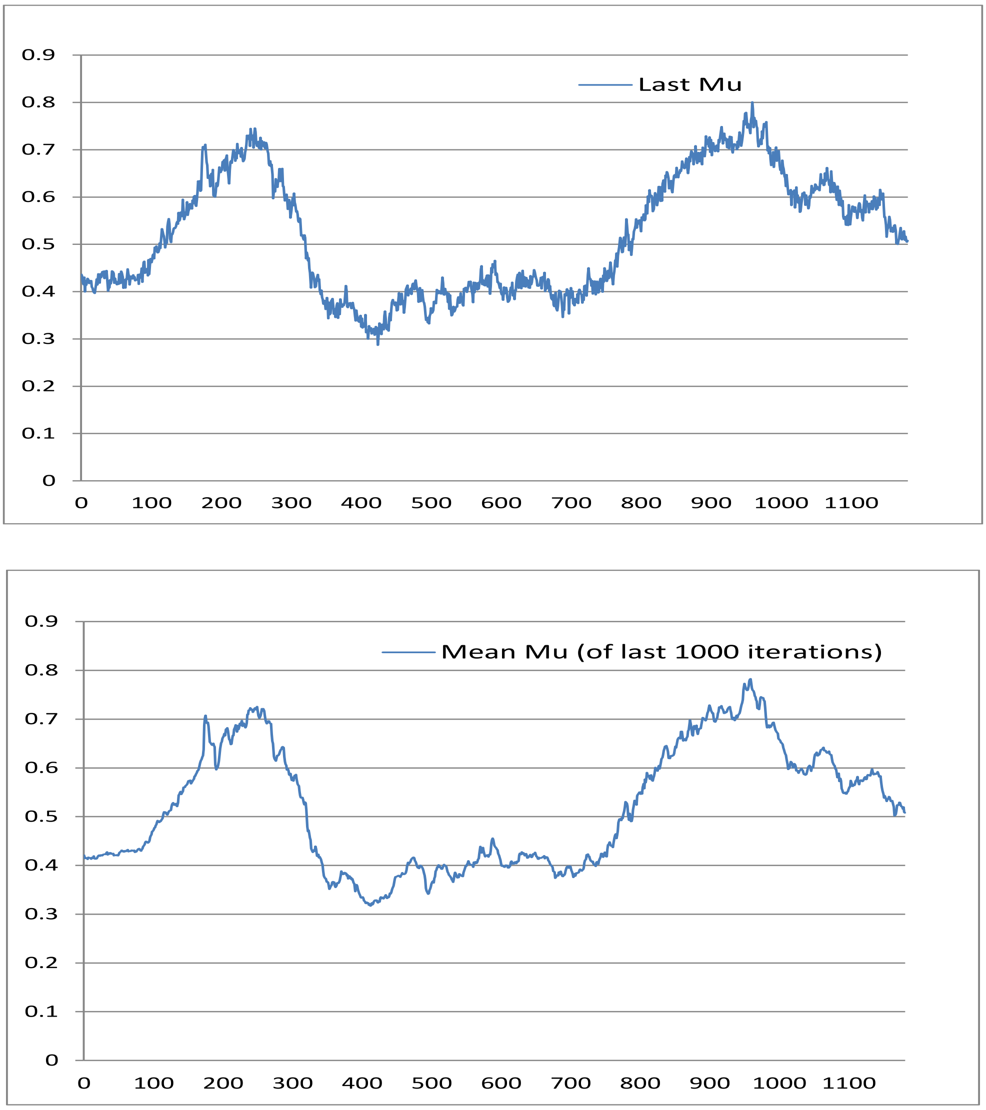
The tables indicate that the value of the parameter γ, while likely positive ($P(\gamma > 0) > 0.95$ for both sets of starting values of the parameters), its expected value is small (around 0.06), indicating the price of soybean oil has a small direct impact on the price of biodiesel. The price of biodiesel seems to be more strongly driven in the short run by factors affecting the path of the latent variables (as reflected by the estimated expected value of β) than by the price of soybean oil. The error terms of the equation of latent variable and that of the price of biodiesel are strongly correlated (the expected value of ρ exceeds 0.90).

The data available for this applications was at a daily frequency. Short run, temporary deviations between the prices of biodiesel and that of its main feedstock (soybean oil) are plausible, even if over the longer run arbitrage behavior by market participants prevents these deviations from being sustained over time. Lower frequency data could be used to better investigate these equilibrium relationships. However, data availability prevents us from pursuing that analysis at this time.

In this study, we focused on whether the price of soybean oil, as a major cost component, is a driver of, or leads changes in the price of biodiesel. Another interesting question would be whether biodiesel, as an important source of new demand for vegetable oils, leads changes in the price of this commodity. This is not the focus of the current study. However, previous research, cited above concluded it is not surprising not to find strong causality relationships between the prices of feedstocks and that of biofuels even if one is present, as the direction and causes of these relationships change over time.

Estimates of the means for the latent variables are shown in Figure 5, which include the estimate from the last iteration (upper panel) and that obtained as the average of the last 100 iterations (lower panel). While the patterns and levels in both panels are similar, the average of the last 100 iterations result in a smoother curve.

Figure 5. Estimates of the latent variable for each *t* with the last iteration (upper panel) and average of the last 100 iterations (bottom panel).

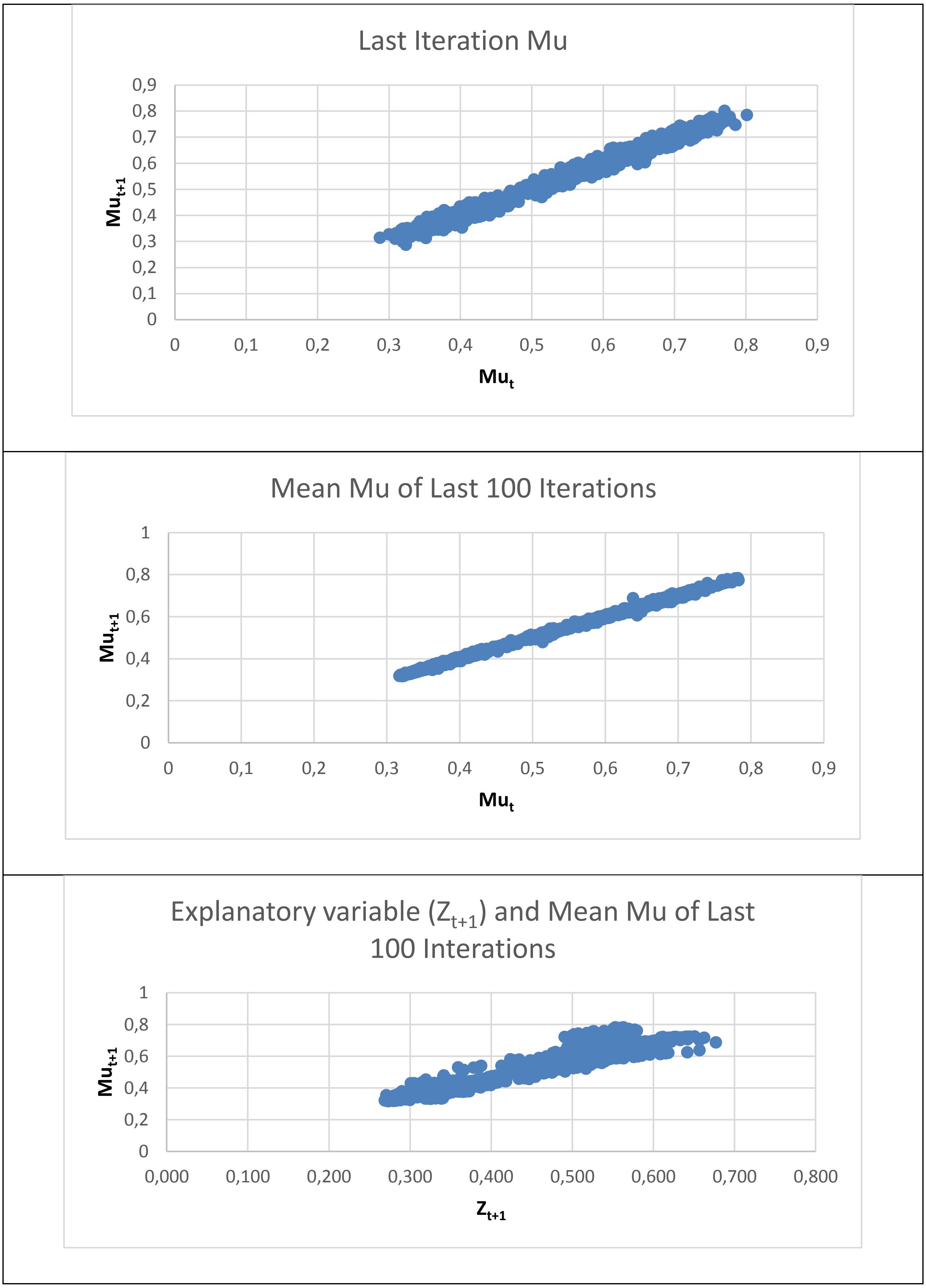


Model Diagnostics

The appropriateness of the model to analyze the data at hand was investigated. In particular, we assess a) whether the linear relationship between the variables, and b) whether the normality of the error terms are reasonable assumptions for this data.

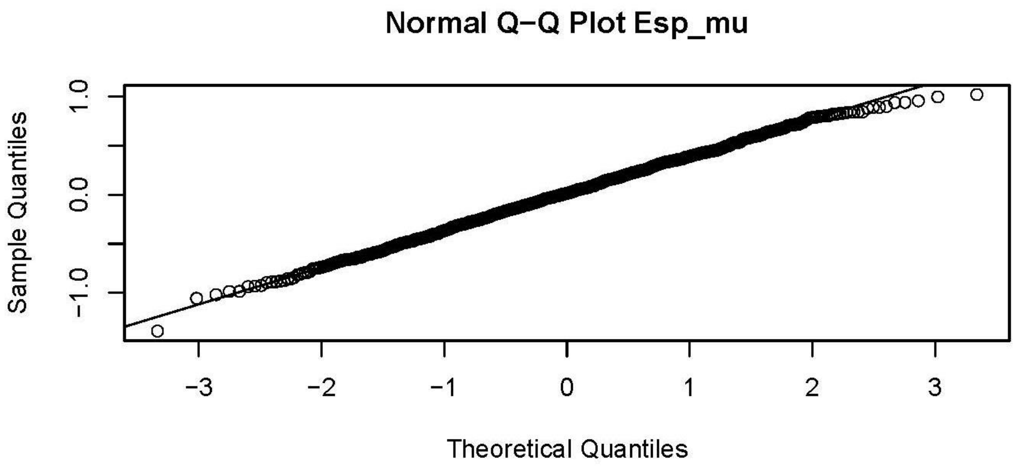
To analyze the linearity assumption between, scatterplots for both the relationship between the explanatory variable Z_{t+1} and the latent variable μ_{t+1} at the same time period, and for consecutive values of the latent variable (μ_t, μ_{t+1}) were constructed. The scatterplots are presented in figure 6, and indicate the linearity assumption is reasonable.

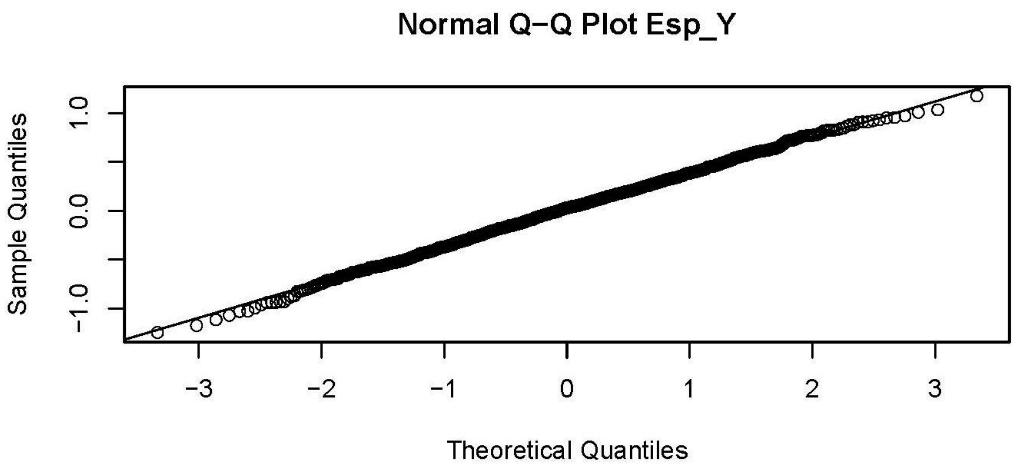
Figure 6. Exploration of the linearity assumption, between the latent variables in consecutive periods for the last iteration (upper panel) and average of the last 100 iteration (middle panel), and between the explanatory and latent variables in the average of the last 100 iterations (bottom panel).



The appropriateness of the normality assumptions in the model for the data at hand was assessed using normal probability plots for both error terms (ε_{t+1}^{y} and ε_{t+1}^{μ}). These plots are presented in Figure 7, and confirm the normality assumptions is reasonably supported by the data.

Figure 7. Exploration of the normality assumption, for the error components ε_{t+1}^{μ} (upper panel) and ε_{t+1}^{y} (lower panel)





8. Conclusions

This work utilized a discrete time return model of finance to analyze whether prices changes of soybean oil, the main feedstock for biodiesel production in the US affect the prices of biodiesel. The model's parameters were estimated using MCMC methods, which were first shown to be able to identify the both the model parameters and the latent variables involved.

Results from this study indicate the price of soybean oil does not have a strong direct impact on the price of biodiesel in the short run, or in a daily basis. Previous studies on the relationships between the price of feedstocks and those of biofuels had mixed results, even if theoretically the relationships should exist. Potential explanations for these puzzles were put forth by Zilberman et al. (2012), who indicated that the directional effect of the price of biofuels on the prices of feedstock depends on the cause of the change in biofuel prices. Therefore, these directional effects may change over time.

While this study analyzed only whether the price of soybean oil drove the price of biodiesel (at a daily frequency). In particular, and given data limitations, it did not use lower frequency to analyze equilibrium relationship, and it did not attempt to analyze whether the price of biodiesel affected that of soybean oil. Both of these analysis could be pursued in other studies.

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Appendix

Key steps in the derivations of the posterior distributions of the parameters

Posterior distribution of E_μ ***conditioned on*** β ***,*** σ_y ***,*** σ_μ ***,*** ρ ***,*** $\underline{\mu}$ ***,*** γ ***,*** Y ***, and*** Z ***:***

$$p\left(E_\mu \middle| \beta,\sigma_y,\sigma_\mu,\rho,\underline{\mu},Y,\gamma\right)$$

$$=\prod_{t=0}^{T-1}\frac{1}{2\pi\sigma_y\sigma_\mu\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\begin{array}{l}\frac{(y_{t+1}-\mu_t)}{\sigma_y^2}-2\rho\left(\frac{y_{t+1}-\mu_t}{\sigma_y}\right)\left(\frac{\mu_{t+1}-E_\mu-\beta(\mu_t-E_\mu)-\gamma Z_{t+1}}{\sigma_\mu}\right)+\\ \left(\frac{\mu_{t+1}-E_\mu-\beta(\mu_t-E_\mu)-\gamma Z_{t+1}}{\sigma_\mu}\right)^2\end{array}\right]\right\}$$

$$\frac{1}{\sqrt{2\pi}E}\exp\left\{-\frac{1}{2}\left(\frac{E_\mu-e}{E}\right)^2\right\}$$

$$\propto\prod_{t=0}^{T-1}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\begin{array}{l}\frac{(y_{t+1}-\mu_t)}{\sigma_y^2}-2\rho\left(\frac{y_{t+1}-\mu_t}{\sigma_y}\right)\left(\frac{\mu_{t+1}-E_\mu-\beta(\mu_t-E_\mu)-\gamma Z_{t+1}}{\sigma_\mu}\right)+\\ \left(\frac{\mu_{t+1}-E_\mu-\beta(\mu_t-E_\mu)-\gamma Z_{t+1}}{\sigma_\mu}\right)^2\end{array}\right]-\frac{1}{2}\left(\frac{E_\mu-e}{E}\right)^2\right\}$$

Let $D_{t+1}=\mu_{t+1}-\beta\mu_t-\gamma Z_{t+1}$ and $C_{t+1}=y_{t+1}-\mu_t$

$$\propto\exp\left\{-\frac{1}{2(1-\rho^2)}\left[-2\rho\sum\frac{C_{t+1}\left(D_{t+1}-E_\mu+\beta E_\mu\right)}{\sigma_\mu\sigma_y}+\sum\left(\frac{D_{t+1}-E_\mu+\beta E_\mu}{\sigma_\mu}\right)^2\right]-\frac{1}{2}\left(\frac{E_\mu-e}{E}\right)^2\right\}$$

$$\propto\exp\left\{-\frac{\rho}{1-\rho^2}\sum\frac{C_{t+1}E_\mu(1-\beta)}{\sigma_\mu\sigma_y}-\frac{1}{2(1-\rho^2)}\sum\left(\frac{E_\mu^2(1-\beta)^2-2D_{t+1}E_\mu(1-\beta)_\mu}{\sigma_\mu^2}\right)-\frac{1}{2}\frac{E_\mu^2}{E^2}+\frac{E_\mu e^2}{E^2}\right\}$$

$$=\exp\left\{\left(-\frac{\rho}{1-\rho^2}\sum\frac{C_{t+1}(1-\beta)}{\sigma_\mu\sigma_y}+\frac{1}{1-\rho^2}\sum\frac{D_{t+1}(1-\beta)}{\sigma_\mu^2}+\frac{e}{E^2}\right)E_\mu-\left(\frac{1}{2(1-\rho^2)}\sum\frac{(1-\beta)^2}{\sigma_\mu^2}+\frac{1}{2E^2}\right)E_\mu^2\right\}$$

Multiplying by -2

$$= \exp\left\{\left(\frac{1}{(1-\rho^2)}\sum\frac{(1-\beta)^2}{\sigma_\mu^2}+\frac{1}{E^2}\right)E_\mu^2 - 2\left(\frac{-\rho}{1-\rho^2}\sum\frac{C_{t+1}(1-\beta)}{\sigma_\mu\sigma_y}+\frac{1}{1-\rho^2}\sum\frac{D_{t+1}(1-\beta)}{\sigma_\mu^2}+\frac{e}{E^2}\right)E_\mu\right\}$$

Let $W = \frac{1}{(1-\rho^2)}\sum\frac{(1-\beta)^2}{\sigma_\mu^2}+\frac{1}{E^2}$, and

$S = \frac{-\rho}{1-\rho^2}\sum\frac{C_{t+1}(1-\beta)}{\sigma_\mu\sigma_y}+\frac{1}{1-\rho^2}\sum\frac{D_{t+1}(1-\beta)}{\sigma_\mu^2}+\frac{e}{E^2}$, the term in the exponent is

$$WE_\mu^2 - 2SE_\mu = W\left(E_\mu^2 - 2\frac{S}{W}E_\mu\right)$$

$$= W\left(E_\mu^2 - 2\frac{S}{W}E_\mu + \left(\frac{S}{W}\right)^2 - \left(\frac{S}{W}\right)^2\right) = W\left(\left(E_\mu - \frac{S}{W}\right)^2 - \left(\frac{S}{W}\right)^2\right)$$

$$\Rightarrow p\left(E_\mu \middle| \beta,\sigma_y,\sigma_\mu,\rho,\underline{\mu},Y,\gamma\right) \propto \exp\left\{-\frac{1}{2W^{-1}}\left(E_\mu - \frac{S}{W}\right)^2\right\}$$

$$\Rightarrow p\left(E_\mu \middle| \beta,\sigma_y,\sigma_\mu,\rho,\underline{\mu},Y,\gamma\right) \sim N\left(\frac{S}{W},W^{-1}\right)$$

Posterior distribution of γ conditioned on β , σ_y , σ_μ , ρ , $\underline{\mu}$, E_μ , Y , and Z :

$$p\left(\beta \middle| E_\mu,\sigma_y,\sigma_\mu,\rho,\underline{\mu},\gamma,Y,Z\right)$$

$$= \prod_{t=0}^{T-1}\frac{1}{2\pi\sigma_y\sigma_\mu\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\begin{array}{l}\frac{(y_{t+1}-\mu_t)}{\sigma_y^2} - 2\rho\left(\frac{y_{t+1}-\mu_t}{\sigma_y}\right)\left(\frac{\mu_{t+1}-E_\mu-\beta(\mu_t-E_\mu)-\gamma Z_{t+1}}{\sigma_\mu}\right)+\\ \left(\frac{\mu_{t+1}-E_\mu-\beta(\mu_t-E_\mu)-\gamma Z_{t+1}}{\sigma_\mu}\right)^2\end{array}\right]\right\}$$

$$\frac{1}{\sqrt{2\pi}F}\exp\left\{-\frac{1}{2}\left(\frac{\beta-f}{F}\right)^2\right\}$$

$$\propto \prod_{t=0}^{T-1}\exp\left\{\frac{2\rho}{2(1-\rho^2)}\left[\begin{array}{l}\left(\frac{y_{t+1}-\mu_t}{\sigma_y}\right)\left(\frac{\mu_{t+1}-E_\mu-\beta(\mu_t-E_\mu)-\gamma Z_{t+1}}{\sigma_\mu}\right)-\\ \frac{1}{2(1-\rho^2)}\left(\frac{\mu_{t+1}-E_\mu-\beta(\mu_t-E_\mu)-\gamma Z_{t+1}}{\sigma_\mu}\right)^2\end{array}\right]-\frac{1}{2}\left(\frac{\beta-f}{F}\right)^2\right\}$$

Let $D_{t+1} = \mu_{t+1} - E_\mu - \gamma Z_{t+1}$ and $C_{t+1} = y_{t+1} - \mu_t$

$$= \exp\left\{ \frac{\rho}{1-\rho^2} \sum \frac{C_{t+1}}{\sigma_y} \frac{D_{t+1} - \beta(\mu_t - E_\mu)}{\sigma_\mu} - \frac{1}{2(1-\rho^2)} \sum \left(\frac{D_{t+1} - \beta(\mu_t - E_\mu)}{\sigma_\mu} \right)^2 - \frac{(\beta - f)^2}{2F^2} \right\}$$

$$\propto \exp\left\{ -\frac{\rho}{1-\rho^2} \sum \frac{C_{t+1}\beta(\mu_t - E_\mu)}{\sigma_y \sigma_\mu} - \frac{1}{2(1-\rho^2)} \sum \left(\frac{(\mu_t - E_\mu)^2 \beta^2 - 2D_{t+1}(\mu_t - E_\mu)\beta}{\sigma_\mu^2} \right) - \frac{\beta^2}{2F^2} + \frac{\beta f}{F^2} \right\}$$

$$= \exp\left\{ \left(\frac{1}{1-\rho^2} \sum \left(\frac{D_{t+1}(\mu_t - E_\mu)}{\sigma_\mu^2} \right) - \frac{\rho}{1-\rho^2} \sum \frac{C_{t+1}(\mu_t - E_\mu)}{\sigma_y \sigma_\mu} + \frac{f}{F^2} \right)\beta - \left(\frac{1}{2(1-\rho^2)} \sum \frac{(\mu_t - E_\mu)^2}{\sigma_\mu^2} + \frac{1}{2F^2} \right)\beta^2 \right\}$$

Let $W = \frac{1}{(1-\rho^2)} \sum \frac{(\mu_t - E_\mu)^2}{\sigma_\mu^2} + \frac{1}{F^2}$, and

$S = \frac{-\rho}{1-\rho^2} \sum \frac{C_{t+1}(\mu_t - E_\mu)}{\sigma_\mu \sigma_y} + \frac{1}{1-\rho^2} \sum \frac{D_{t+1}(\mu_t - E_\mu)}{\sigma_\mu^2} + \frac{f}{F^2}$, the term in the exponent is

$$-\frac{1}{2}(W\beta^2 - 2S\beta) = -\frac{1}{2}W\left(\beta^2 - 2\frac{S}{W}\beta\right) = -\frac{W}{2}\left(\beta^2 - 2\frac{S}{W}\beta + \left(\frac{S}{W}\right)^2 - \left(\frac{S}{W}\right)^2\right)$$

$$= -\frac{W}{2}\left(\left(\beta - \frac{S}{W}\right)^2 - \left(\frac{S}{W}\right)^2\right) = -\frac{W}{2}\left(\beta - \frac{S}{W}\right)^2 + \frac{W}{2}\left(\frac{S}{W}\right)^2$$

Then, going back to the exponential equation above we have

$$\exp\left\{ -\frac{1}{2W^{-1}}\left(\beta - \frac{S}{W}\right)^2 \right\} \exp\left\{ \frac{W}{2}\left(\frac{S}{W}\right)^2 \right\}$$

$$\propto \exp\left\{ -\frac{1}{2W^{-1}}\left(\beta - \frac{S}{W}\right)^2 \right\}$$

$$\Rightarrow p\left(\beta \middle| E_\mu, \sigma_y, \sigma_\mu, \rho, \underline{\mu}, Y, \gamma\right) \propto \exp\left\{ -\frac{1}{2W^{-1}}\left(\beta - \frac{S}{W}\right)^2 \right\}$$

$$\Rightarrow p\left(\beta \middle| E_\mu, \sigma_y, \sigma_\mu, \rho, \underline{\mu}, Y, \gamma\right) \sim N\left(\frac{S}{W}, W^{-1}\right)$$

Posterior distribution of γ ***conditioned on*** β , σ_y , σ_μ , ρ , $\underline{\mu}$, E_μ , Y ***, and*** Z ***:***

$$p\left(\gamma \middle| E_\mu,\beta,\sigma_y,\sigma_\mu,\rho,\underline{\mu},Y\right)$$

$$=\prod_{t=0}^{T-1}\frac{1}{2\pi\sigma_y\sigma_\mu\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\begin{array}{l}\frac{\left(y_{t+1}-\mu_t\right)}{\sigma_y^2}-2\rho\left(\frac{y_{t+1}-\mu_t}{\sigma_y}\right)\left(\frac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right)+\\\left(\frac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right)^2\end{array}\right]\right\}$$

$$\frac{1}{\sqrt{2\pi}R}\exp\left\{-\frac{1}{2}\left(\frac{\gamma-r}{R}\right)^2\right\}$$

$$\propto\prod_{t=0}^{T-1}\exp\left\{\frac{1}{2\left(1-\rho^2\right)}\left[\begin{array}{l}2\rho\left(\frac{y_{t+1}-\mu_t}{\sigma_y}\right)\left(\frac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right)-\\\left(\frac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right)^2\end{array}\right]-\frac{1}{2}\left(\frac{\gamma-r}{R}\right)^2\right\}$$

Let $C_{t+1}=y_{t+1}-\mu_t$ and $D_{t+1}=\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)$. Then

$$\propto\prod_{t=0}^{T-1}\exp\left\{\frac{1}{2\left(1-\rho^2\right)}\left[\begin{array}{l}2\rho\left(\frac{y_{t+1}-\mu_t}{\sigma_y}\right)\left(\frac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right)-\\\left(\frac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right)^2\end{array}\right]-\frac{1}{2}\left(\frac{\gamma-r}{R}\right)^2\right\}$$

$$=\prod_{t=0}^{T-1}\exp\left\{\frac{1}{2\left(1-\rho^2\right)}\left[\begin{array}{l}2\rho\left(\frac{C_{t+1}}{\sigma_y}\right)\left(\frac{D_{t+1}-\gamma Z_{t+1}}{\sigma_\mu}\right)-\\\left(\frac{D_{t+1}-\gamma Z_{t+1}}{\sigma_\mu}\right)^2\end{array}\right]-\frac{1}{2}\left(\frac{\gamma-r}{R}\right)^2\right\}$$

$$\propto\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left(2\rho\sum\frac{C_{t+1}Z_{t+1}\gamma}{\sigma_y\sigma_\mu}+\sum\frac{Z_{t+1}^2\gamma^2}{\sigma_\mu^2}-2\sum\frac{D_{t+1}Z_{t+1}\gamma}{\sigma_\mu^2}\right)-\frac{\gamma^2}{2R^2}+\frac{r\gamma}{R^2}\right\}$$

$$=\exp\left\{\left(\sum\frac{D_{t+1}Z_{t+1}}{\sigma_\mu^2\left(1-\rho^2\right)}-\frac{\rho}{\left(1-\rho^2\right)}\sum\frac{C_{t+1}Z_{t+1}}{\sigma_y\sigma_\mu}+\frac{r}{R^2}\right)\gamma-\left(\frac{1}{2\left(1-\rho^2\right)}\sum\frac{Z_{t+1}^2}{\sigma_\mu^2}+\frac{1}{2R^2}\right)\gamma^2\right\}$$

Let $W=\frac{1}{\left(1-\rho^2\right)}\sum\frac{Z_{t+1}^2}{\sigma_\mu^2}+\frac{1}{R^2}$, and $S=\frac{-\rho}{1-\rho^2}\sum\frac{C_{t+1}Z_{t+1}}{\sigma_\mu\sigma_y}+\frac{1}{1-\rho^2}\sum\frac{D_{t+1}Z_{t+1}}{\sigma_\mu^2}+\frac{r}{R^2}$, then the equation can be expressed as

$$= \exp\left\{-\frac{1}{2}\left(W\gamma^2 - 2S\gamma\right)\right\}$$

$$= \exp\left\{-\frac{W}{2}\left(\gamma^2 - 2\frac{S}{W}\gamma + \left(\frac{S}{W}\right)^2 - \left(\frac{S}{W}\right)^2\right)\right\}$$

$$= \exp\left\{-\frac{W}{2}\left(\gamma - \frac{S}{W}\right)^2 - W\left(\frac{S}{W}\right)^2\right\}$$

$$\propto \exp\left\{-\frac{W}{2}\left(\gamma - \frac{S}{W}\right)^2\right\}$$

$$\Rightarrow p\left(\gamma \middle| E_\mu, \beta, \sigma_y, \sigma_\mu, \rho, \underline{\mu}, Y\right) \propto \exp\left\{-\frac{1}{2W^{-1}}\left(\gamma - \frac{S}{W}\right)^2\right\}$$

$$\Rightarrow p\left(\gamma \middle| E_\mu, \beta, \sigma_y, \sigma_\mu, \rho, \underline{\mu}, Y\right) \sim N\left(\frac{S}{W}, W^{-1}\right)$$

Posterior distribution of $\underline{\mu}$ conditioned on β , σ_y , σ_μ , ρ , γ , E_μ , Y , and Z :

$\underline{\mu}_0$

$$p\left(\mu_0 \middle| E_\mu, \beta, \sigma_y, \sigma_\mu, \rho, \underline{\mu}, \gamma, Y, Z\right)$$

$$= \frac{1}{2\pi\sigma_y\sigma_\mu\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\begin{array}{l}\frac{\left(y_1-\mu_0\right)^2}{\sigma_y^2} - 2\rho\left(\frac{y_1-\mu_0}{\sigma_y}\right)\left(\frac{\mu_1 - E_\mu - \beta\left(\mu_0 - E_\mu\right) - \gamma Z_1}{\sigma_\mu}\right) + \\ \left(\frac{\mu_1 - E_\mu - \beta\left(\mu_0 - E_\mu\right) - \gamma Z_1}{\sigma_\mu}\right)^2\end{array}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\begin{array}{l}\frac{-2y_1\mu_0 + \mu_0^2}{\sigma_y^2} - \frac{2\rho}{\sigma_y\sigma_\mu} - y_1\beta\mu_0 - \mu_0\left(\mu_1 - E_\mu - \gamma Z_1\right) + \mu_0^2\beta - \mu_0\beta E_\mu \\ -\frac{2}{\sigma_\mu^2}\left(\mu_1\mu_0\beta - \beta E_\mu\mu_0\right) + \frac{1}{\sigma_\mu^2}\left(\beta^2\left(\mu_0 - E_\mu\right)^2 + 2\beta\left(\mu_0 - E_\mu\right)\gamma Z_1 + \gamma Z^2\right)\end{array}\right]\right\}$$

$$\propto \exp\left\{ \begin{array}{l} -\frac{1}{2(1-\rho^2)}\left(\frac{1}{\sigma_y^2}-\frac{2\rho\beta}{\sigma_y\sigma_\mu}+\frac{\beta}{\sigma_\mu^2}\right)\mu_0^2 - \\ 2\left(\frac{y_1}{\sigma_y^2}-\frac{\rho y_1\beta}{\sigma_y\sigma_\mu}-\frac{\rho}{\sigma_y\sigma_\mu}(\mu_1-E_\mu-\gamma Z_1)-\frac{\rho\beta E_\mu}{\sigma_y\sigma_\mu}+\frac{\mu_1\beta}{\sigma_\mu^2}-\frac{\beta E_\mu}{\sigma_\mu^2}+\frac{\beta^2 E_\mu}{\sigma_\mu^2}-\frac{\beta\gamma Z_1}{\sigma_\mu^2}\right)\mu_0 \end{array} \right\}$$

Let $W=\frac{1}{\sigma_y^2}-\frac{2\rho\beta}{\sigma_y\sigma_\mu}+\frac{\beta}{\sigma_\mu^2}$, and

$S=\frac{y_1}{\sigma_y^2}-\frac{\rho y_1\beta}{\sigma_y\sigma_\mu}-\frac{\rho}{\sigma_y\sigma_\mu}(\mu_1-E_\mu-\gamma Z_1)-\frac{\rho\beta E_\mu}{\sigma_y\sigma_\mu}+\frac{\mu_1\beta}{\sigma_\mu^2}-\frac{\beta E_\mu}{\sigma_\mu^2}+\frac{\beta^2 E_\mu}{\sigma_\mu^2}-\frac{\beta\gamma Z_1}{\sigma_\mu^2}$, then the equation can be expressed as

$$=\exp\left\{-\frac{1}{2(1-\rho^2)}\left(W\mu_0^2-2S\mu_0\right)\right\}$$

$$=\exp\left\{-\frac{1}{2(1-\rho^2)}W\left(\mu_0^2-2\frac{S}{W}\mu_0+\left(\frac{S}{W}\right)^2-\left(\frac{S}{W}\right)^2\right)\right\}$$

$$=\exp\left\{-\frac{1}{2(1-\rho^2)}W\left(\left(\mu_0-\frac{S}{W}\right)^2-\left(\frac{S}{W}\right)^2\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2(1-\rho^2)W^{-1}}\left(\mu_0-\frac{S}{W}\right)^2\right\}$$

$$\Rightarrow p\left(\mu_0 \middle| E_\mu,\beta,\sigma_y,\sigma_\mu,\rho,\underline{\mu},\gamma,Y,Z\right)\propto \exp\left\{-\frac{1}{2(1-\rho^2)W^{-1}}\left(\mu_0-\frac{S}{W}\right)^2\right\}$$

$$\Rightarrow p\left(\mu_0 \middle| E_\mu,\beta,\sigma_y,\sigma_\mu,\rho,\underline{\mu},\gamma,Y,Z\right)\sim N\left(\frac{S}{W},\frac{1-\rho^2}{W}\right)$$

$\underline{\mu}_T$

$p\left(\mu_T \middle| E_\mu,\beta,\sigma_y,\sigma_\mu,\rho,\underline{\mu},\gamma,Y,Z\right)$

$$=\frac{1}{2\pi\sigma_y\sigma_\mu\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\begin{array}{l} \frac{(y_T-\mu_{T-1})^2}{\sigma_y^2}-2\rho\left(\frac{y_T-\mu_{T-1}}{\sigma_y}\right)\left(\frac{\mu_T-E_\mu-\beta(\mu_{T-1}-E_\mu)-\gamma Z_T}{\sigma_\mu}\right)+ \\ \left(\frac{\mu_T-E_\mu-\beta(\mu_{T-1}-E_\mu)-\gamma Z_T}{\sigma_\mu}\right)^2 \end{array} \right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\begin{array}{l}\frac{-2\rho y_T\mu_T}{\sigma_y\sigma_\mu}+\frac{2\rho\mu_{T-1}\mu_T}{\sigma_y\sigma_\mu}+\frac{\mu_T^2-2\mu_T E_\mu+E_\mu^2}{\sigma_\mu^2}-\\ \frac{2\mu_T\left(\beta\left(\mu_{T-1}-E_\mu\right)+\gamma Z_T\right)}{\sigma_\mu^2}+\frac{2E_\mu\left(\beta\left(\mu_{T-1}-E_\mu\right)+\gamma Z_T\right)}{\sigma_\mu^2}\end{array}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{\mu_T^2}{\sigma_\mu^2}-2\left(\frac{\rho y_T}{\sigma_y\sigma_\mu}-\frac{\rho\mu_{T-1}}{\sigma_y\sigma_\mu}+\frac{E_\mu}{\sigma_\mu^2}+\frac{\beta\mu_{T-1}}{\sigma_\mu^2}-\frac{\beta E_\mu}{\sigma_\mu^2}+\frac{\gamma Z_T}{\sigma_\mu^2}\right)\mu_T\right]\right\}$$

Let $W=\frac{1}{\sigma_y^2}$, and $S=\frac{\rho y_T}{\sigma_y\sigma_\mu}-\frac{\rho\mu_{T-1}}{\sigma_y\sigma_\mu}+\frac{E_\mu}{\sigma_\mu^2}+\frac{\beta\mu_{T-1}}{\sigma_\mu^2}-\frac{\beta E_\mu}{\sigma_\mu^2}+\frac{\gamma Z_T}{\sigma_\mu^2}$, then the equation can be expressed as

$$=\exp\left\{-\frac{1}{2(1-\rho^2)}\left(W\mu_T^2-2S\mu_T\right)\right\}$$

$$=\exp\left\{-\frac{1}{2(1-\rho^2)}W\left(\mu_T^2-2\frac{S}{W}\mu_T+\left(\frac{S}{W}\right)^2-\left(\frac{S}{W}\right)^2\right)\right\}$$

$$=\exp\left\{-\frac{1}{2(1-\rho^2)}W\left(\left(\mu_T-\frac{S}{W}\right)^2-\left(\frac{S}{W}\right)^2\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2(1-\rho^2)W^{-1}}\left(\mu_T-\frac{S}{W}\right)^2\right\}$$

$$\Rightarrow p\left(\mu_T\left|E_\mu,\beta,\sigma_y,\sigma_\mu,\rho,\underline{\mu},\gamma,Y,Z\right.\right)\propto \exp\left\{-\frac{1}{2(1-\rho^2)W^{-1}}\left(\mu_T-\frac{S}{W}\right)^2\right\}$$

$$\Rightarrow p\left(\mu_T\left|E_\mu,\beta,\sigma_y,\sigma_\mu,\rho,\underline{\mu},\gamma,Y,Z\right.\right)\sim N\left(\frac{S}{W},\frac{1-\rho^2}{W}\right)$$

$\underline{\mu}_t$ for $0<t<T$

$$p\left(\mu_t \middle| E_\mu, \beta, \sigma_y, \sigma_\mu, \rho, \underline{\mu}, \gamma, Y, Z\right)$$

$$= \frac{1}{2\pi\sigma_y\sigma_\mu\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2\left(1-\rho^2\right)} \left[\begin{array}{l} \frac{\left(y_{t+1}-\mu_t\right)^2}{\sigma_y^2} - 2\rho\left(\frac{y_{t+1}-\mu_t}{\sigma_y}\right)\left(\frac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right) + \\ \left(\frac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right)^2 \end{array} \right] \right\}$$

$$\frac{1}{2\pi\sigma_y\sigma_\mu\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2\left(1-\rho^2\right)} \left[\begin{array}{l} \frac{\left(y_t-\mu_{t-1}\right)^2}{\sigma_y^2} - 2\rho\left(\frac{y_t-\mu_{t-1}}{\sigma_y}\right)\left(\frac{\mu_t-E_\mu-\beta\left(\mu_{t-1}-E_\mu\right)-\gamma Z_t}{\sigma_\mu}\right) + \\ \left(\frac{\mu_t-E_\mu-\beta\left(\mu_{t-1}-E_\mu\right)-\gamma Z_t}{\sigma_\mu}\right)^2 \end{array} \right] \right\}$$

$$\propto \exp\left\{ -\frac{1}{2\left(1-\rho^2\right)} \left[\begin{array}{l} \frac{-2\mu_t y_{t+1}}{\sigma_y^2} + \frac{\mu_t^2}{\sigma_y^2} - 2\rho\left(\frac{-y_{t+1}\beta\mu_t - \mu_t\mu_{t+1} + \mu_t E_\mu + \mu_t^2\beta - \mu_t\beta E_\mu + \gamma\mu_t Z_{t+1}}{\sigma_\mu\sigma_y} - \right) + \\ -\frac{2\left(\mu_{t+1}-E_\mu\right)\beta\left(\mu_t-E_\mu\right)}{\sigma_\mu^2} + \frac{\beta^2\left(\mu_t-E_\mu\right)^2}{\sigma_\mu^2} + \frac{2\beta\left(\mu_t-E_\mu\right)\gamma Z_{t+1}}{\sigma_\mu^2} + \frac{\left(\gamma Z_{+1}\right)^2}{\sigma_\mu^2} \\ -\frac{2\rho y_t\mu_t}{\sigma_\mu\sigma_y} + \frac{2\rho\mu_t\mu_{t-1}}{\sigma_\mu\sigma_y} + \frac{\mu_t^2}{\sigma_\mu^2} - \frac{2\mu_t E_\mu}{\sigma_\mu^2} + \frac{E_\mu^2}{\sigma_\mu^2} - \frac{2\mu_t\beta\left(\mu_{t-1}-E_\mu\right)}{\sigma_\mu^2} - \frac{2\mu_t\gamma Z_t}{\sigma_\mu^2} \end{array} \right] \right\}$$

$$\propto \exp\left\{ -\frac{1}{2\left(1-\rho^2\right)} \left[\begin{array}{l} -2\left(\begin{array}{l} \frac{y_{t+1}}{\sigma_y^2} - \frac{\rho y_{t+1}\beta}{\sigma_\mu\sigma_y} - \frac{\rho\mu_{t+1}}{\sigma_\mu\sigma_y} + \frac{\rho E_\mu}{\sigma_\mu\sigma_y} - \frac{\rho\beta E_\mu}{\sigma_\mu\sigma_y} + \frac{\rho\gamma Z_{t+1}}{\sigma_\mu\sigma_y} + \frac{\left(\mu_{t+1}-E_\mu\right)\beta}{\sigma_\mu^2} \\ + \frac{\beta^2 E_\mu}{\sigma_\mu^2} - \frac{\beta\gamma Z_{t+1}}{\sigma_\mu^2} + \frac{\rho y_t}{\sigma_\mu\sigma_y} - \frac{\rho\mu_{t-1}}{\sigma_\mu\sigma_y} + \frac{E_\mu}{\sigma_\mu^2} + \frac{\beta\left(\mu_{t-1}-E_\mu\right)}{\sigma_\mu^2} + \frac{\gamma Z_t}{\sigma_\mu^2} \end{array} \right)\mu_t \\ + \left(\frac{1}{\sigma_y^2} - \frac{2\rho\beta}{\sigma_\mu\sigma_y} + \frac{\beta^2}{\sigma_\mu^2} + \frac{1}{\sigma_\mu^2}\right)\mu_t^2 \end{array} \right] \right\}$$

Let $W = \frac{1}{\sigma_y^2} - \frac{2\rho\beta}{\sigma_\mu\sigma_y} + \frac{\beta^2}{\sigma_\mu^2} + \frac{1}{\sigma_\mu^2}$, and

$$S = \frac{y_{t+1}}{\sigma_y^2} - \frac{\rho y_{t+1}\beta}{\sigma_\mu\sigma_y} - \frac{\rho\mu_{t+1}}{\sigma_\mu\sigma_y} + \frac{\rho E_\mu}{\sigma_\mu\sigma_y} - \frac{\rho\beta E_\mu}{\sigma_\mu\sigma_y} + \frac{\rho\gamma Z_{t+1}}{\sigma_\mu\sigma_y} + \frac{\left(\mu_{t+1}-E_\mu\right)\beta}{\sigma_\mu^2} + \frac{\beta^2 E_\mu}{\sigma_\mu^2} - \frac{\beta\gamma Z_{t+1}}{\sigma_\mu^2} + \frac{\rho y_t}{\sigma_\mu\sigma_y} - \frac{\rho\mu_{t-1}}{\sigma_\mu\sigma_y} + \frac{E_\mu}{\sigma_\mu^2} + \frac{\beta\left(\mu_{t-1}-E_\mu\right)}{\sigma_\mu^2} + \frac{\gamma Z_t}{\sigma_\mu^2}$$

, then the equation can be expressed as

$$= \exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left(W\mu_t^2 - 2S\mu_t\right)\right\}$$

$$= \exp\left\{-\frac{1}{2\left(1-\rho^2\right)}W\left(\mu_t^2 - 2\frac{S}{W}\mu_t + \left(\frac{S}{W}\right)^2 - \left(\frac{S}{W}\right)^2\right)\right\}$$

$$= \exp\left\{-\frac{1}{2\left(1-\rho^2\right)}W\left(\left(\mu_t - \frac{S}{W}\right)^2 - \left(\frac{S}{W}\right)^2\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2\left(1-\rho^2\right)W^{-1}}\left(\mu_t - \frac{S}{W}\right)^2\right\}$$

$$\Rightarrow p\left(\mu_t \middle| E_\mu, \beta, \sigma_y, \sigma_\mu, \rho, \underline{\mu}, \gamma, Y, Z\right) \propto \exp\left\{-\frac{1}{2\left(1-\rho^2\right)W^{-1}}\left(\mu_t - \frac{S}{W}\right)^2\right\}$$

$$\Rightarrow p\left(\mu_t \middle| E_\mu, \beta, \sigma_y, \sigma_\mu, \rho, \underline{\mu}, \gamma, Y, Z\right) \sim N\left(\frac{S}{W}, \frac{1-\rho^2}{W}\right)$$

Posterior joint distribution of $\left(\rho, \sigma_\mu\right)$ ***conditioned on*** β **,** σ_y**,** $\underline{\mu}$ **,**γ**,** E_μ **,** Y **, *and*** Z **:**

For this distribution it is convenient to use the transformation (Jacquier, Ploson and Rossi, 1994)

$$\begin{cases} \phi_\mu = \sigma_\mu \rho \\ \omega_\mu = \sigma_\mu^2\left(1-\rho^2\right) \end{cases}.$$

The joint prior for $\left(\phi_\mu, \omega_\mu\right)$ is

$$\begin{cases} \phi_\mu \middle| \omega_\mu \sim N\left(0, \frac{1}{2}\omega_\mu\right) \\ \omega_\mu \sim IG\left(a, b\right) \end{cases}$$

We will use the following facts from the transformation to obtain the posterior

$$\rho = \frac{\phi_\mu}{\sigma_\mu},\ \omega_\mu + \phi_\mu^2 = \sigma_\mu^2,\ \rho = \frac{\phi_\mu}{\sqrt{\omega_\mu + \phi_\mu^2}},\ \sigma_\mu = \sqrt{\omega_\mu},\ 1-\rho^2 = \frac{\sqrt{\omega_\mu}}{\sqrt{\omega_\mu + \phi_\mu^2}},\ \frac{\rho}{\sigma_\mu} = \frac{\phi_\mu}{\omega_\mu + \phi_\mu^2}$$

$$p\left(\phi_\mu,\omega_\mu \middle| E_\mu,\beta,\sigma_y,\underline{\mu},Y,Z\right)$$

$$\propto \prod_{t=0}^{T-1}\frac{1}{2\pi\sigma_y\sigma_\mu\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\begin{array}{l}\frac{\left(y_{t+1}-\mu_t\right)^2}{\sigma_y^2}-2\rho\left(\frac{y_{t+1}-\mu_t}{\sigma_y}\right)\left(\frac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right)+\\ \left(\frac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right)^2\end{array}\right]\right\}$$

$$\left(\frac{1}{\omega_\mu}\right)^{a+1}\exp\left[-\frac{1}{b}\frac{1}{\omega_\mu}\right]\frac{1}{\sqrt{2\pi\frac{1}{2}\omega_\mu}}\exp\left[-\frac{\left(\phi_\mu-0\right)^2}{2\frac{1}{2}\omega_\mu}\right]$$

Let $C_t=\dfrac{y_{t+1}-\mu_t}{\sigma_y}$, and $D_t=\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}$

$$\propto \prod_{t=0}^{T-1}\frac{1}{2\pi\sigma_y\sqrt{\omega_\mu}}\exp\left[-\frac{\omega_\mu+\phi_\mu^2}{2\omega_\mu}\left\{C_t^2-2C_tD_t\frac{\phi_\mu}{\omega_\mu+\phi_\mu^2}+D_t^2\frac{1}{\omega_\mu+\phi_\mu^2}\right\}\right]\left(\frac{1}{\omega_\mu}\right)^{a+1}\exp\left[-\frac{1}{b}\frac{1}{\omega_\mu}\right]\frac{1}{\sqrt{\pi\omega_\mu}}\exp\left[-\frac{\phi_\mu^2}{\omega_\mu}\right]$$

$$\propto\left(\frac{1}{2\pi\sigma_y\sqrt{\omega_\mu}}\right)^T\exp\left[-\frac{1}{2\omega_\mu}\left\{\sum C_t^2\left(\omega_\mu+\phi_\mu^2\right)-2\left(\sum C_tD_t\right)\phi_\mu+\left(\sum D_t^2\right)\right\}\right]$$

$$\left(\frac{1}{\omega_\mu}\right)^{a+1}\exp\left[-\frac{1}{b\omega_\mu}\right]\frac{1}{\sqrt{\pi\omega_\mu}}\exp\left[-\frac{\phi_\mu^2}{\omega_\mu}\right]$$

$$\propto\left(\frac{1}{\omega_\mu}\right)^{\frac{T}{2}+a+1+\frac{1}{2}}\exp\left[-\frac{1}{2\omega_\mu}\left\{\sum C_t^2\left(\omega_\mu+\phi_\mu^2\right)-2\left(\sum C_tD_t\right)\phi_\mu+\left(\sum D_t^2\right)\right\}\right]\exp\left[-\frac{1}{b\omega_\mu}\right]\exp\left[-\frac{\phi_\mu^2}{\omega_\mu}\right]$$

$$\propto\left(\frac{1}{\omega_\mu}\right)^{\frac{T}{2}+a+\frac{3}{2}}\exp\left[\left(-\frac{1}{2\omega_\mu}\sum C_t^2-\frac{1}{\omega_\mu}\right)\phi_\mu^2+\left(\frac{1}{\omega_\mu}\sum C_tD_t\right)\phi_\mu\right]\exp\left[-\frac{1}{2\omega_\mu}\sum D_t^2-\frac{1}{b\omega_\mu}\right]$$

Let

$$x=\exp\left[\left(-\frac{1}{2\omega_\mu}\sum C_t^2-\frac{1}{\omega_\mu}\right)\phi_\mu^2+\left(\frac{1}{\omega_\mu}\sum C_tD_t\right)\phi_\mu\right]$$

$$=\exp\left[-\frac{1}{2\omega_\mu}\left[\left(\sum C_t^2+2\right)\phi_\mu^2-2\left(\sum C_tD_t\right)\phi_\mu\right]\right]$$

Let $S=\sum_{t=1}^{T-1}C_tD_t$ and $W=\sum_{t=1}^{T-1}C_t^2+2$

$$x = \exp\left[-\frac{1}{2\omega_\mu}\left\{W\phi_\mu^2 - 2S\phi_\mu\right\}\right]$$

$$= \exp\left[-\frac{1}{2\frac{\omega_\mu}{W}}\left\{\phi_\mu^2 - 2\frac{S}{W}\phi_\mu\right\}\right]$$

$$= \exp\left[-\frac{1}{2\frac{\omega_\mu}{W}}\left\{\phi_\mu^2 - 2\frac{S}{W}\phi_\mu + \frac{S^2}{W^2} - \frac{S^2}{W^2}\right\}\right]$$

$$= \exp\left[\frac{1}{2}\frac{S^2}{\omega_\mu W}\right]\exp\left[-\frac{1}{2\frac{\omega_\mu}{W}}\left(\phi_\mu - \frac{S}{W}\right)^2\right]$$

$$= \exp\left[\frac{1}{\omega_\mu}\frac{S^2}{2W}\right]\sqrt{2\pi\frac{\omega_\mu}{W}}\,\frac{1}{\sqrt{2\pi\frac{\omega_\mu}{W}}}\exp\left[-\frac{\left(\phi_\mu - \frac{S}{W}\right)^2}{2\frac{\omega_\mu}{W}}\right]$$

The second half represents a normal density, with mean $\frac{S}{W}$ and variance $\frac{\omega_\mu}{W}$, the form is

$\frac{1}{\sqrt{2\pi\frac{\omega_\mu}{W}}}\exp\left[-\frac{\left(\phi_\mu - \frac{S}{W}\right)^2}{2\frac{\omega_\mu}{W}}\right]$. Thus, the posterior $p\left(\phi_\mu, \omega_\mu \middle| E_\mu, \beta, \sigma_y, \underline{\mu}, Y, Z\right)$ is proportional to

$$\propto \left(\frac{1}{\omega_\mu}\right)^{\frac{T}{2}+a+\frac{3}{2}-\frac{1}{2}}\sqrt{2\pi\frac{1}{W}}\exp\left[-\frac{1}{2\omega_\mu}\sum D_t^2 - \frac{1}{b\omega_\mu}\right]\exp\left[\frac{1}{\omega_\mu}\frac{S^2}{2W}\right]N\left(\frac{S}{W}, \frac{\omega_\mu}{W}\right)$$

$$\propto \left(\frac{1}{\omega_\mu}\right)^{\frac{T}{2}+a+1}\sqrt{2\pi\frac{1}{W}}\exp\left[-\frac{1}{\omega_\mu}\left\{\frac{\sum D_t^2}{2} + \frac{1}{b}\right\} + \frac{1}{\omega_\mu}\frac{S^2}{2W}\right]N\left(\frac{S}{W}, \frac{\omega_\mu}{W}\right)$$

$$\propto \left(\frac{1}{\omega_\mu}\right)^{\frac{T}{2}+a+1}\sqrt{2\pi\frac{1}{W}}\exp\left[-\frac{1}{\omega_\mu}\left\{\frac{\sum D_t^2}{2} + \frac{1}{b} - \frac{S^2}{2W}\right\}\right]N\left(\frac{S}{W}, \frac{\omega_\mu}{W}\right)$$

$$IG\left(\frac{T}{2}+a,\frac{1}{\frac{\sum D_t^2}{2}+\frac{1}{b}-\frac{S^2}{2W}}\right)N\left(\frac{S}{W},\frac{\omega_\mu}{W}\right)$$

Therefore, the posterior of $\left(\phi_\mu,\omega_\mu\right)$ is

$$\begin{cases}\phi_\mu\middle|\omega_\mu \sim N\left(\frac{S}{W},\frac{\omega_\mu}{W}\right)\\ \omega_\mu \sim IG\left(\frac{T}{2}+a,\frac{1}{\frac{\sum_{t=1}^{T} D_t^2}{2}+\frac{1}{b}-\frac{S^2}{2W}}\right)\end{cases}$$

Posterior distribution of σ_y ***conditioned on*** β ***,*** σ_μ ***,*** ρ ***,*** $\underline{\mu}$ ***,*** E_μ ***,*** γ ***,*** Y ***, and*** Z ***:***

$$p\left(\sigma_y\middle|E_\mu,\beta,,\sigma_\mu,\rho,\gamma,\underline{\mu},Y,Z\right)$$

$$=\prod_{t=0}^{T-1}\frac{1}{2\pi\sigma_y\sigma_\mu\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\begin{array}{l}\frac{\left(y_{t+1}-\mu_t\right)}{\sigma_y^2}-2\rho\left(\frac{y_{t+1}-\mu_t}{\sigma_y}\right)\left(\frac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right)+\\ \left(\frac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right)^2\end{array}\right]\right\}$$

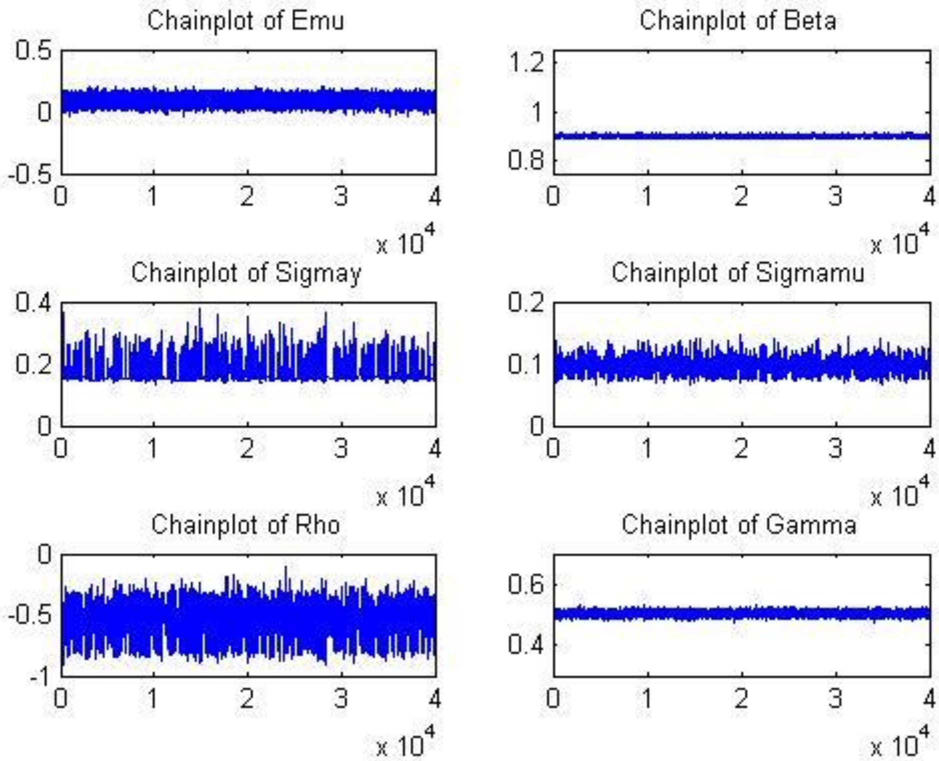
$$\left(\frac{1}{\sigma_y^2}\right)^{\alpha-1}\frac{\beta^\alpha\exp\left(-\frac{1}{\beta}\frac{1}{\sigma_y^2}\right)}{\Gamma\left(\alpha\right)}$$

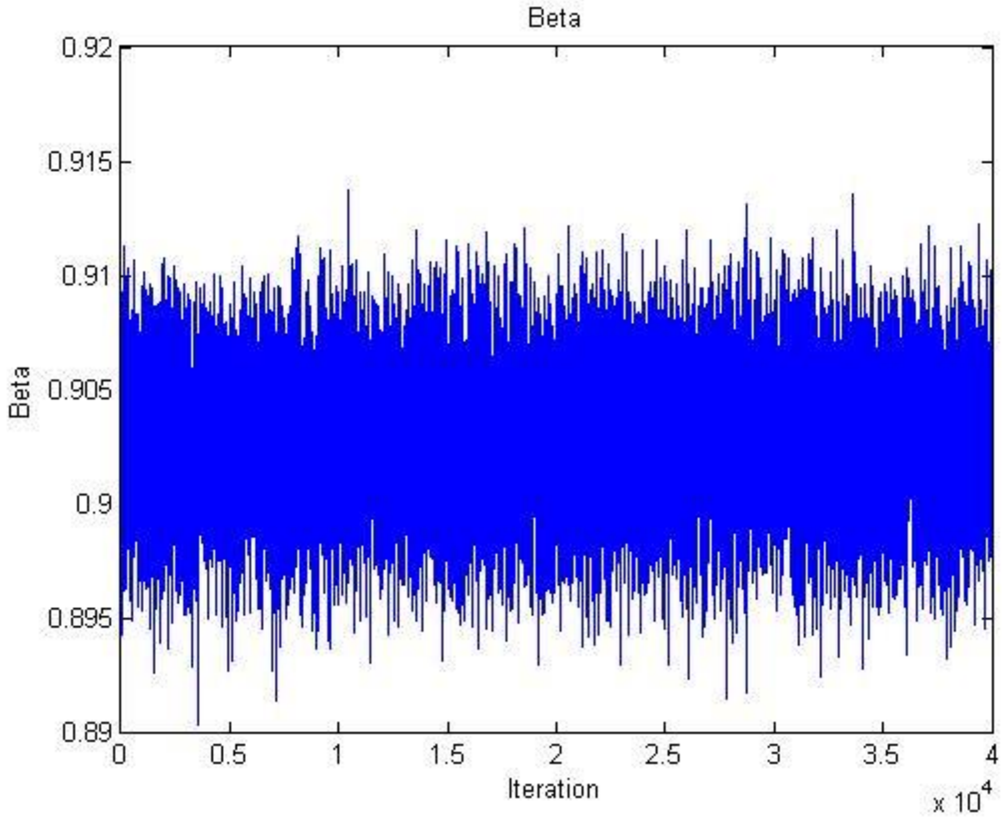
$$\propto\left(\frac{1}{2\pi\sigma_y\sigma_\mu\sqrt{1-\rho^2}}\right)^T\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\sum\left[\begin{array}{l}\frac{\left(y_{t+1}-\mu_t\right)^2}{\sigma_y^2}-2\rho\left(\frac{y_{t+1}-\mu_t}{\sigma_y}\right)\left(\frac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right)+\\ \left(\frac{\mu_{t+1}-E_\mu-\beta\left(\mu_t-E_\mu\right)-\gamma Z_{t+1}}{\sigma_\mu}\right)^2\end{array}\right]\right\}$$

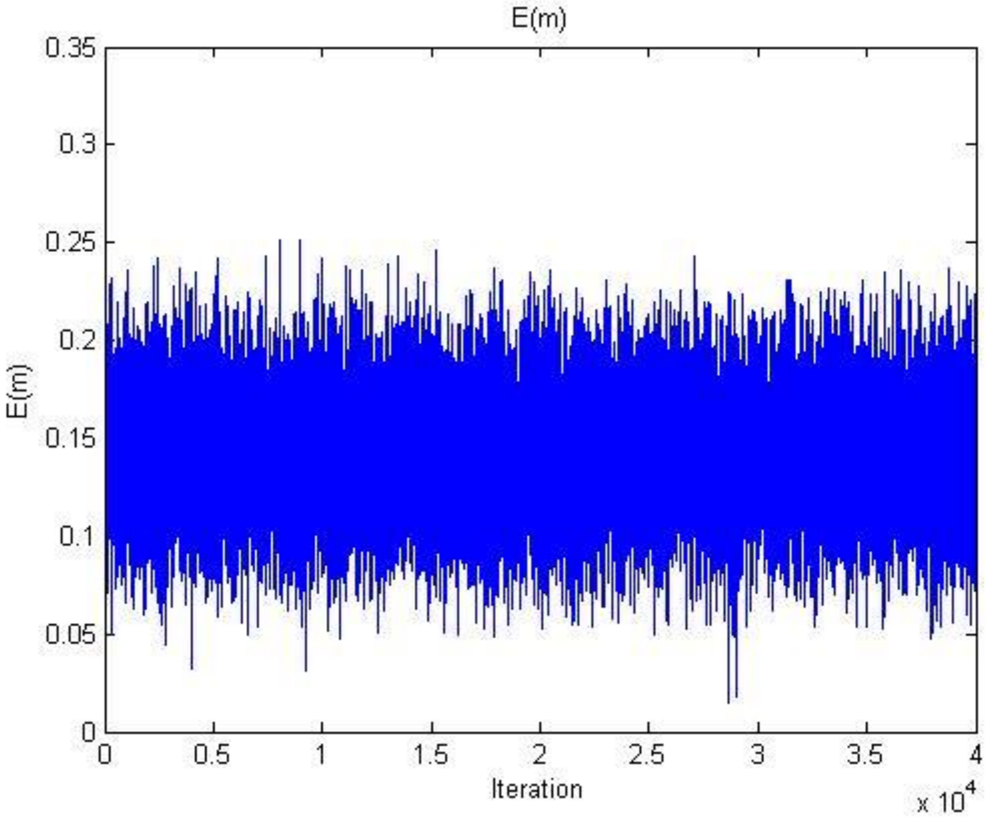
$$\left(\frac{1}{\sigma_y^2}\right)^{\alpha-1}\exp\left(-\frac{1}{\beta}\sigma_y^2\right)$$

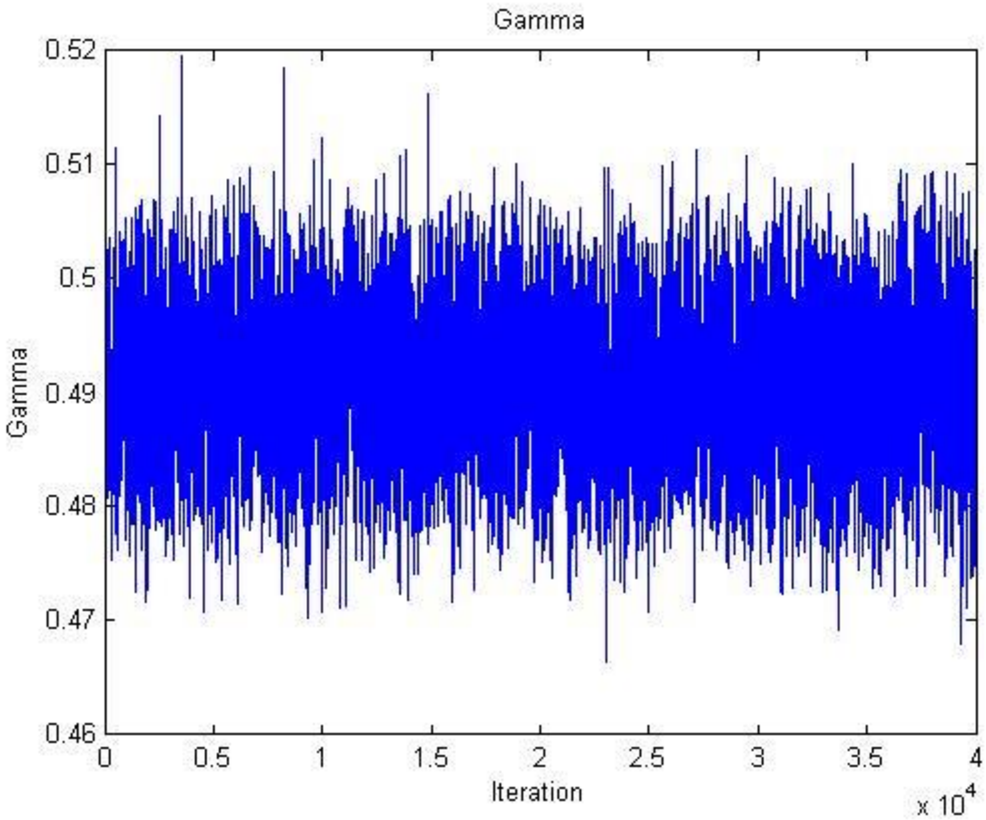
There is no closed form solution for the distribution of σ_y. The Metropolis-Hasting algorithm will be used to obtain samples from this distribution.

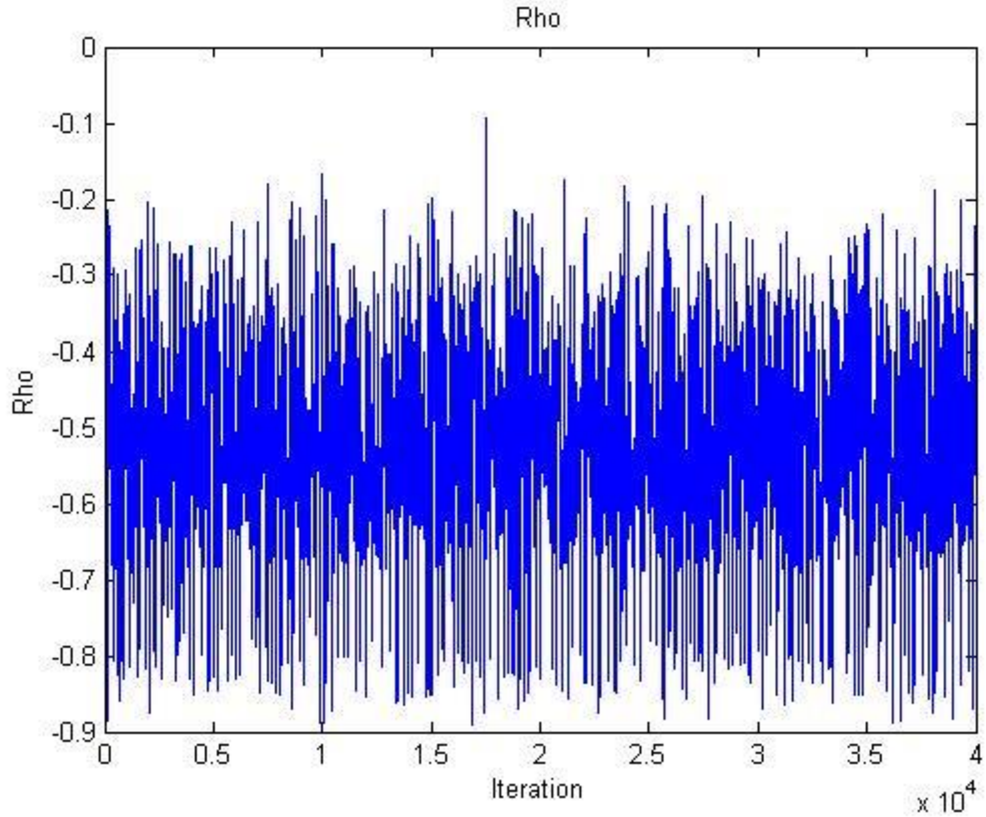
The figures below show the chain of the simulated parameters excluding the burn-in period. The plots show that the chains for all parameters have converged and are stable. The posterior means, estimated as the averages for each chain after the burn-in period (shown in Table 2) are close to the true parameters used to simulate the data.

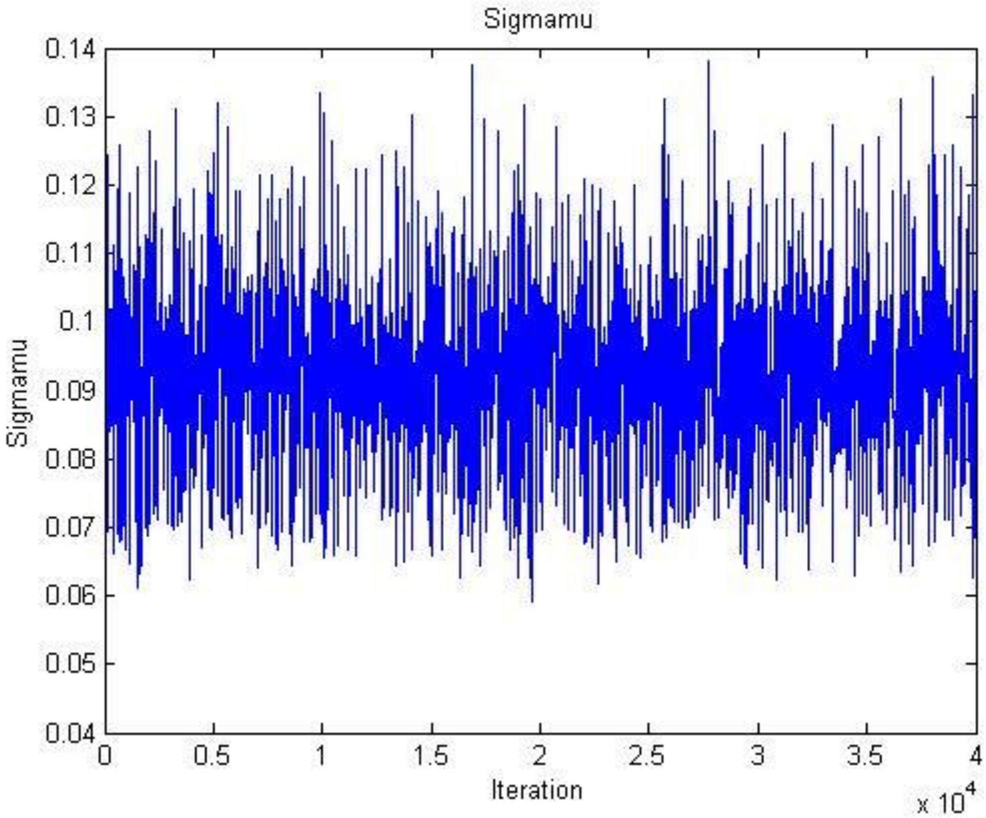


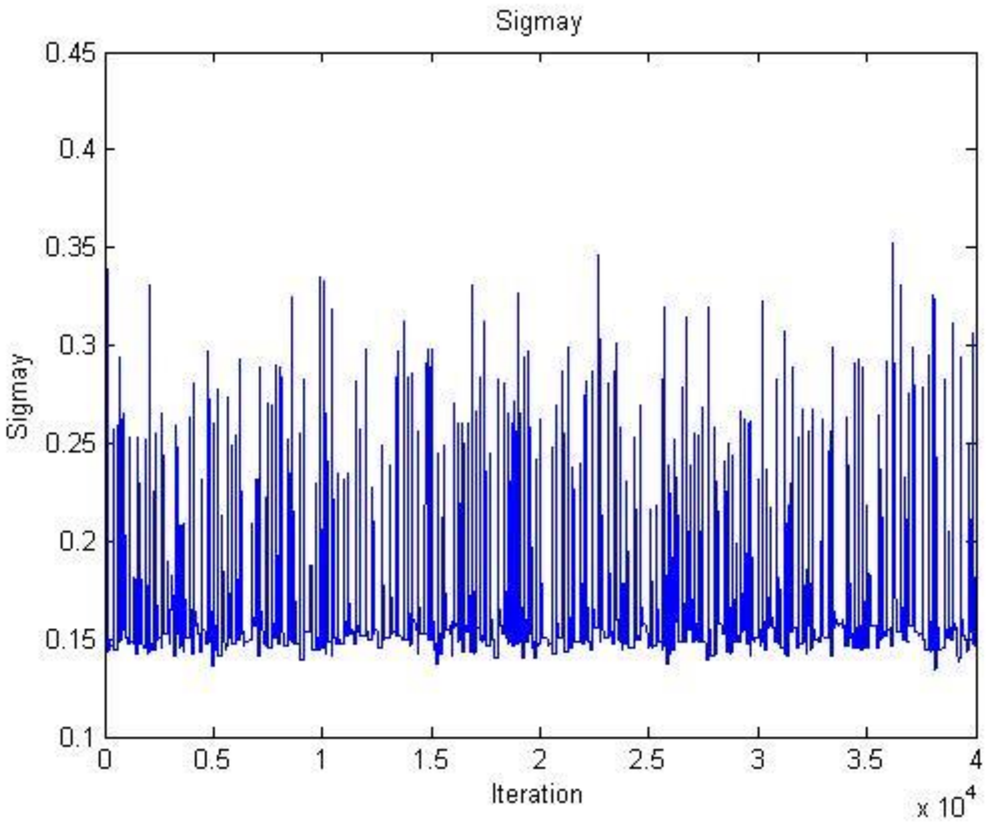

Beta
0.92
0.915
0.91
0.905
0.9
0.895
0.89
Beta
0
0.5
1
1.5
2
2.5
3
3.5
4
Iteration
x 10^4


E(m)
E(m)
Iteration
x 10^4

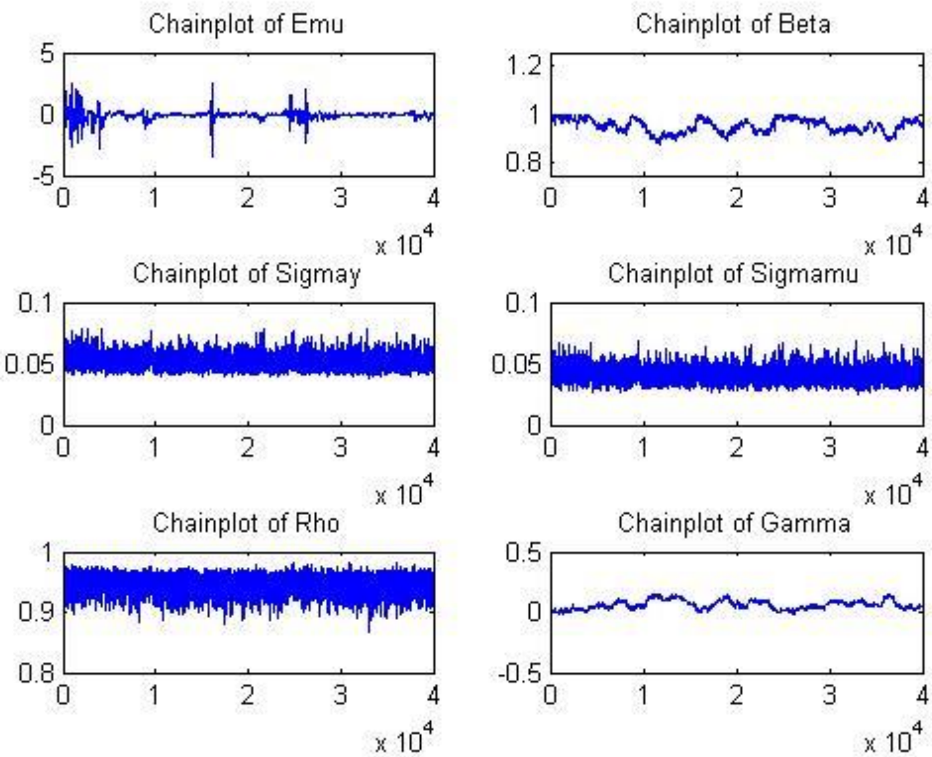

Gamma
Gamma
Iteration
x 10^4


Rho
Rho
Iteration
x 10^4


Sigmamu
Sigmamu
Iteration
x 10^4



Plots of the chains for the parameters using the second set of starting values.



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Facultad de Ciencias Económicas y de Administración

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